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Dynamic coordination among heterogeneous agents ☆

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ABSTRACT

We study a dynamic model of coordination with timing frictions and payoff heterogeneity. There is a unique equilibrium, characterized by thresholds that determine the choices of each type of agent. We characterize equilibrium for the limiting cases of vanishing timing frictions and vanishing shocks to fundamentals. A lot of conformity emerges: despite payoff heterogeneity, agents' equilibrium thresholds partially coincide as long as a set of beliefs that would make this coincidence possible exists. However, the equilibrium thresholds never fully coincide. In case of vanishing frictions, the economy behaves almost as if all agents were equal to an average type. Conformity is not inefficient. In the efficient solution, agents follow others even more often.

1. Introduction

When deciding between Facebook and Google+, a consumer will take into account what others have been choosing but also her own tastes. In a problem of debt roll-over, both coordination motives and an individual's appetite for risk have to be considered. Firms' investment decisions depend on the expected demand for their goods, which, in turn, depends on whether other firms will be investing and on idiosyncratic factors that affect the demand for a particular product. Similarly, adopting a new technology may not be the best decision if others in the production chain will keep working with an old technology but heterogeneity in agents' productivity might also play an important role in this problem. In all these settings, both payoff complementarities and idiosyncratic features of preferences or technologies are important for an agent's choice.

Strategic complementarities induce players to try to do the same thing. In a dynamic setting, that means following what others are doing now and what they are likely to choose in the future. However, the idiosyncratic component of payoffs might push agents in different directions. This paper studies the interplay of complementarities and heterogeneity in payoffs in a dynamic setting.

In order to study this question we consider a dynamic environment with timing frictions as in Frankel and Pauzner (2000). Agents make a binary choice between two actions (say joining Facebook or not). Agents' instantaneous utility flow depends on an exogenously moving fundamental (which captures the intrinsic quality of Facebook), on how many others are in the network and on idiosyncratic tastes. Agents get opportunities to revise their behavior (join or leave Facebook) according to a Poisson clock, which can be seen as an attention friction modeled in a reduced-form way.

We first show there is a unique rationalizable equilibrium where agents of a given type play according to a threshold that depends on the total number of agents in a network and on the exogenous fundamental. We then obtain analytical results for the limiting cases of vanishing shocks and vanishing frictions, and provide an analytical characterization of the equilibrium thresholds in a tractable case with linear utility. Last, we solve the planner's problem to understand the inefficiencies related to payoff heterogeneity that arise in equilibrium.

In equilibrium, each type of agent joins the network if the exogenous fundamental (θ) is larger than a threshold, which is a function of the fraction of agents in the network (n). In the tractable limiting cases, a lot of conformity arises. Different types will always play the same strategy for some values on n unless their preferences are so heterogeneous that there is no set of (arbitrary) beliefs that would induce them to play according to the same threshold. Agents' choices are more similar for intermediate values of n, when there is more heterogeneity in their behavior – and more

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dispersion of beliefs in a neighborhood around the threshold. However, from a social point of view, there is not enough conformity. The region where agents play different strategies in the planner's solution is smaller than the analogous region in the decentralized equilibrium.

In case of vanishing frictions, although agents play according to different strategies, the economy behaves almost as if all agents were identical and equal to an average type (again, unless agents' preferences are so heterogeneous that no set of beliefs could induce conformity).

This paper builds on the model of Frankel and Pauzner (2000). They base their analysis on a model of sectorial choice as in Matsuyama (1991). Similar frameworks have been used to study neighborhood choices (Frankel and Pauzner, 2002), carry trades and speculation (Plantin and Shin, 2006), speculative attacks (Daniëls, 2009), business cycles (Frankel and Burdzy, 2005; Guimaraes and Machado, forthcoming) and efficiency in settings with network externalities (Guimaraes and Pereira, 2016).[1]

The paper is related to the literature on coordination in games with strategic complementarities. With complete information and no shocks, these games might exhibit multiple self-fulfilling equilibria. Carlsson and Van Damme (1993), Morris and Shin (1998) and Frankel et al. (2003) have shown that a unique equilibrium arise in a static environment in which fundamentals are not common knowledge and agents have idiosyncratic information about them (the so called global games). Frankel and Pauzner (2000) and Burdzy et al. (2001) show that a small amount of shocks in a dynamic model (with no private information) yields similar results. The relation between both literatures is discussed in Morris (2014).[2] In a related contribution, Herrendorf et al. (2000) show that if there is enough heterogeneity and a continuum of types, there is a unique equilibrium even in a dynamic setting with complete information.

Applied work employing the global games methodology has often considered heterogeneous populations in static coordination games.[3] Our results can be used in applied settings where dynamic coordination and heterogeneity are both important.

The paper is also related to the literature on network externalities where strategic complementarities arise from consumption externalities.[4] As in this paper, the optimal action for one agent depends on her expectations about other agents' choices. However, most of that literature makes ad-hoc assumptions regarding which equilibrium is played.[5] One important exception is Argenziano (2008). She studies welfare in a model with differentiated networks in a static global-game model and highlights two sources of inefficiencies: agents give too much importance to their own idiosyncratic tastes and firms with the larger network charge a higher price. Both effects contribute to make the network "too balanced". This paper complements her work by studying coordination among heterogeneous agents in a dynamic setting.[6]

The efficiency results here differ from those in models with information externalities that generate herd behavior (e.g., Bikhchandani et al., 1992). There, agents follow others too much from a social point of view. Here, conformity of behavior arises because of preferences, not through learning, and the efficient solution features agents following others even more often.

2. The model

There is a continuum of infinitely-lived agents indexed by $i \in [0, 1]$. Time is continuous and agents discount the future at rate ρ. There are two possible actions $a_i \in \{0, 1\}$, but agents cannot switch from one to another at will. They receive chances to revise their actions according to a Poisson process with arrival rate δ, and stay committed to this choice until the arrival of another opportunity. This timing friction might represent an attention friction of consumers or firms, a machine break-up in an environment with a choice between two technologies or maturity of debt in a model of debt runs.

The flow payoff an agent gets from either action depends on fundamentals, on her idiosyncratic preferences and on the actions of others (there are strategic complementarities). Let n be the proportion of agents choosing action 1. Strategic complementarities can arise owing to either one-sided externalities or two-sided externalities: either the payoff of choosing action 0 is independent of the amount of agents making the same choice, but the payoff of choosing 1 is increasing in n (as in Matsuyama, 1991); or both actions become more appealing the larger is the proportion of agents taking them (as in Argenziano, 2008); or flow-payoffs from both actions can be increasing in n, but the difference in payoffs is also monotonically increasing in n (as in Guimaraes and Machado, forthcoming).

We denote agent i's relative flow-payoff of choosing action 1 by $\pi_{q(i)}(\theta, n)$, where $\theta \in \mathbb{R}$ denotes the fundamentals of the economy, $n \equiv \int_0^1 a_i di$ is the fraction of agents currently committed to action 1 and $q(i) \in \mathcal{Q} = \{1, \ldots, Q\}$ is agent i's type. All functions $\pi_q(.)$ are continuously differentiable and strictly increasing in both arguments. If we let α_q denote the mass of type-q agents in the population and n_q the proportion of type-q agents currently playing 1, n can be written as $n = \sum_{q=1}^{Q} \alpha_q n_q$.

An agent who receives a chance to revise her choice at time τ will choose $a_i = 1$ whenever

$$\mathbb{E} \int_\tau^\infty e^{-(\rho+\delta)(t-\tau)} \pi_{q(i)}(\theta_t, n_t) dt > 0$$

and $a_i = 0$ if the inequality is reversed. The expected discounted payoff takes into account only the states in which the agent believes she will still be committed to her action ($e^{-\delta(t-\tau)}$ expresses the probability of not receiving a revising opportunity between τ and t).

We further assume that payoff functions $\pi_q(.)$ are such that there are dominance regions for all types of agents. For each type, there is a region in the $\mathbb{R} \times [0, 1]$ space where choosing action 0 is a dominant action, and a region in which choosing action 1 is a dominant action. More specifically, for any given initial n, there is a sufficiently low level of fundamentals at which an agent prefers to play 0 even if she expects all others to play 1 when they get a chance to revise their actions, and there is a sufficiently high level of fundamentals such that it is preferable to play 1 even if no one else is expected to choose so in the future.[7]

Let P_q be the boundary of the upper dominance region of a type-q agent, i.e., the curve on which such agent is indifferent between

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[1] The model of currency attacks in Guimaraes (2006) and the model of debt runs in He and Xiong (2012) have similar timing frictions.

[2] See also Morris and Shin (2003).

[3] Examples include heterogeneity in wealth (Goldstein and Pauzner, 2004); roles (Goldstein, 2005); risk aversion and consumption profile (Guimaraes and Morris, 2007); externalities from production (Sakovics and Steiner, 2012); and financial health (Choi, 2014).

[4] Seminal papers in this literature include Katz and Shapiro (1985, 1986). See Shy (2011) for a survey.

[5] For instance, Katz and Shapiro (1986) assume that whenever there are multiple equilibria in the model, the Pareto-superior outcome is achieved.

[6] See also Ambrus and Argenziano (2009).

[7] Formally, we can define the lower dominance region boundary for type q as a curve $O_q(n_0)$ satisfying

$$\mathbb{E}\left[\int_0^\infty e^{-(\rho+\delta)t} \pi_q(\theta_t, n_t^\uparrow) | \theta_0 = O_q\right] = 0,$$

where $n_t^\uparrow = 1 - (1 - n_0)e^{-\delta t}$. Likewise, we can define the upper dominance region boundary as the curve $P_q(n_0)$ satisfying

$$\mathbb{E}\left[\int_0^\infty e^{-(\rho+\delta)t} \pi_q(\theta_t, n_t^\downarrow) | \theta_0 = P_q\right] = 0,$$

where $n_t^\downarrow = n_0 e^{-\delta t}$.

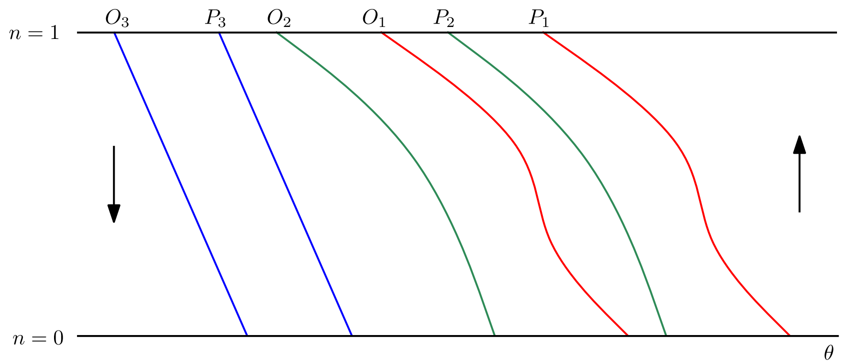

Fig. 1. Dominance regions: an example.

the two actions if she believes everyone after her will choose 0 (P stands for *pessimistic* about the proportion of agents playing 1 in the future). This boundary is downward sloping: since $\pi_q(\theta, n)$ is increasing in θ and n, a higher n today means that the value of θ needed to make agents indifferent between the two actions is smaller. At the other extreme, let O_q be the boundary of the lower dominance region for a type-q player, that is, the curve on which this type of agent is indifferent between the two actions under the belief that everyone will choose 1 when they get the chance (O stands for *optimistic*). This curve is also downward sloping. Fig. 1 shows an example of dominant regions.

2.1. Unique equilibrium

Suppose θ is constant. When θ lies either to the right of all upper dominance regions boundaries, or to the left of all lower dominance regions boundaries, there is only one rationalizable action. Nevertheless, there always exists a subset of the state space with equilibrium multiplicity.[8]

However, when there are shocks to θ, the equilibrium is unique for any amount of heterogeneity. Proposition 1 presents this result. The following lemma is key for the demonstration.

Lemma 1. *Given agents' actions at a given t, the dynamics of n_t depends on each $n_{q,t}$, $q \in \mathcal{Q}$, only through n_t.*

Proof. Fix $\{a_{i,t}\}_{i\in[0,1]}$. Let β_q be the fraction of the type-q population whose strategy prescribes playing 1 at time t. The dynamics of $n_{q,t}$ is given by:

$$\frac{\partial n_{q,t}}{\partial t} = \delta\left(\beta_q - n_{q,t}\right). \tag{1}$$

Eq. (1) captures that a type-q agent whose strategy prescribes playing 1 and is currently playing 0 will switch to action 1 when she receives an opportunity to revise her choice. Likewise, every type-q agent whose strategy prescribes playing 0 and who has previously chosen 1 will switch to action 0 at the first opportunity.

Using the fact that $n = \sum_{q=1}^{Q} \alpha_q n_{q,t}$, we have that the dynamics of n_t is given by:

$$\frac{\partial n_t}{\partial t} = \delta\left[\sum_{q=1}^{Q} \alpha_q \beta_q - n_t\right]. \quad \square$$

Lemma 1 will allow us to deal with this problem in a two-dimensional space. Given agents' choices at time t, we only need to consider the *aggregate* mass of agents currently committed to action 1 in order to understand the dynamics of the system. One could expect this dynamics to depend on the proportion of each type of agent currently locked in each option, but due to the assumption of a Poisson process for the arrival of opportunities to switch actions, that is not true. It suffices to know the aggregate n_t and agents' choices to compute $\partial n_t/\partial t$, no matter the current $\{n_q\}_{q\in\mathcal{Q}}$.

Intuitively, consider that all individuals drawn by the Poisson clock at time t are automatically assigned to option 0 but have the chance to choose 1. At every instant, n_t decreases at a rate δn_t (due to the individuals assigned to option 0), but owing to the choices of agents whose strategies prescribe playing 1, n_t also increases at a rate $\delta \sum_{q=1}^{Q} \alpha_q \beta_q$ (since the law of large numbers holds and the Poisson parameter is the same across groups).

Proposition 1. *Suppose θ follows a Brownian motion with drift μ and variance $\sigma^2 > 0$. There is a unique equilibrium characterized by downward sloping thresholds $\left(Z_q^*\right)_{q\in\mathcal{Q}}$ in the $\mathbb{R} \times [0, 1]$ space, such that*

$$a_{i,t} = \begin{cases} 1 & \text{if } \theta_t > Z^*_{q(i)}(n_t) \\ 0 & \text{if } \theta_t < Z^*_{q(i)}(n_t) \end{cases}$$

That is, each agent i called upon acting at time t plays 1 when to the right and 0 when to the left of $Z^*_{q(i)}$.

Proof. See Appendix B.1. $\square$

The proof of equilibrium uniqueness employs a strategy of iterative elimination of strictly dominated strategies, starting from the dominance regions. Even if these regions are very remote, making it unlikely that the fundamentals will reach one of them before an agent receives another chance to revise her action, the existence of such regions triggers an iterative contagion effect until there is a single rationalizable strategy left (for each type of agent). The basic intuition is as follows: an agent at any point on the boundary of her upper dominance region is indifferent between actions 0 or 1 under the assumption that, at all future dates, all other agents will choose 0. But once shocks to fundamentals are introduced, she knows there is a positive probability that fundamentals will reach regions in which it is dominant for some agents to play 1 (while she is still committed to her choice).[9] Thus, she cannot hold the belief that others will play 0 under any circumstances. The most pessimistic belief she can hold is that agents will play 0 whenever it is not strictly dominated to do so, and under this new (a bit more optimistic) belief, there is another (smaller) level of fundamentals that makes such agent indifferent between the two actions. Extending this reasoning to all following rounds and employing an

[8] Herrendorf et al. (2000) shows that in a similar environment with no shocks and a continuum of types, multiplicity is ruled out if there is a sufficient amount of heterogeneity.

[9] Notice timing frictions are key for the proof. However, the uniqueness result still holds as $\delta \to \infty$.

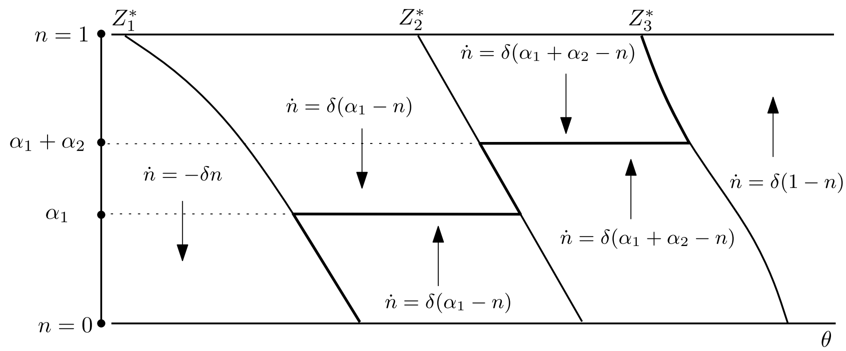

Fig. 2. Dynamics ($Q = 3$).

analogous procedure starting from the lower dominance regions yields a unique rationalizable equilibrium. An interesting aspect of this result – which was demonstrated by Frankel and Pauzner (2000) for the case of identical individuals – is that uniqueness of equilibrium can be achieved even with vanishing shocks to fundamentals (that is, in the limit as $\mu, \sigma \to 0$). Multiplicity of equilibrium in this environment do not survive the introduction of the smallest amount of shocks.

The unique equilibrium is characterized by thresholds for each type of agent, to the right of which these agents play 1, and to the left of which they play 0. Depending on the initial value of n, these strategies imply an upward or downward path for n_t. Fig. 2 exemplifies the dynamics around the equilibrium for a case with three types of agents. $\partial n_t/\partial t$ is computed as in Lemma 1.

3. Limiting cases

We now restrict our attention to situations with two types, $Q = 2$, and analyze two limiting cases: vanishing shocks to fundamentals and vanishing timing frictions. Assume $q(i) = \overline{q}\ \forall i \in [0, \alpha]$ and $q(i) = \underline{q}\ \forall i \in (\alpha, 1]$, i.e., there is a mass α of type-$\overline{q}$ agents in the economy and a mass $1 - \alpha$ of type-$\underline{q}$ agents. Denote their payoff functions, respectively, by $\overline{\pi}(\theta, n)$ and $\underline{\pi}(\theta, n)$, and to save notation let $\overline{P} \equiv P_{\overline{q}}$, $\underline{P} \equiv P_{\underline{q}}$, $\overline{O} \equiv O_{\overline{q}}$, and $\underline{O} \equiv O_{\underline{q}}$ be the dominance regions boundaries for the two types. We assume that for any pair (θ, n), $\overline{\pi}(\theta, n) > \underline{\pi}(\theta, n)$, that is, type-$\overline{q}$ agents have a higher relative instantaneous payoff of choosing action 1 in every state.

The next lemma, based on Burdzy et al. (1998), characterizes agents' beliefs on the equilibrium threshold and is key for the results of the paper.

Lemma 2. *Suppose agents play according to two arbitrary (downward sloping and Lipschitz) thresholds $\overline{Z}(n) < \underline{Z}(n)$ for all n in some interval (n^1, n^2). Consider a point (θ, n) with $n \in (n^1, n^2)$ and either $\theta = \underline{Z}(n)$ or $\theta = \overline{Z}(n)$. As $\mu, \sigma \to 0$, the time it takes for the system to bifurcate either up or down converges to zero. Moreover, the probabilities of an upward or a downward bifurcation are computed as follows:*

(i) *Consider a point (θ, n) with $\theta = \overline{Z}(n)$.*

$$P(up) = \begin{cases} 0 & \text{if } n \geq \alpha \\ 1 - \dfrac{n}{\alpha} & \text{if } n < \alpha \end{cases}$$

and $P(down) = 1 - P(up)$.

(ii) *Consider a point (θ, n) with $\theta = \underline{Z}(n)$.*

$$P(up) = \begin{cases} \dfrac{1-n}{1-\alpha} & \text{if } n > \alpha \\ 1 & \text{if } n \leq \alpha \end{cases}$$

and $P(down) = 1 - P(up)$.

Proof. See Appendix B.2. □

Fig. 3 shows the dynamics around the two types' thresholds in case they do not intersect (computed as in the proof of Lemma 1) and the implied bifurcation probabilities along the thresholds (computed as in Lemma 2). The idea behind Lemma 2 is that, considering an initial point exactly on an agent's threshold, the probability of the system going up or down depends on the speed of increase or decrease of n at each side of the threshold. Intuitively, once the economy has headed off in one direction, it does not revert to $\overline{Z}$ or $\underline{Z}$, since thresholds are downward sloping and shocks to fundamentals are small, but will it start going up or down? That depends on the realization of the Brownian motion in a tiny time span and on the speed of decrease and increase of n at each side of the threshold that pull the economy away from the (downward sloping) threshold.

A few examples help to illustrate the result in Lemma 2. Consider an agent called upon revising her action when the economy is at p_1 in Fig. 3. As $n = 0$, a small negative shock pushing θ slightly to the left will make no difference (n cannot decrease anymore), while a small positive shock to θ will lead high type agents to choose action 1, so agents believe that n will increase with probability one. An agent at point p_2 holds the opposite belief but for a different reason: both to the left and to the right of $\overline{Z}$, n is decreasing, so the agent assigns probability one to n heading towards zero. Last, look at point p_3 in Fig. 3. A small negative shock to θ means that all high-type agents who get the chance will play 1, but all low-types will play 0. Since there are more agents currently committed to 1 than agents willing to choose 1 (because $n > \alpha$), n decreases in that region at a rate $\delta(n - \alpha)$. A small positive shock, though, would make every agent willing to switch to action 1, hence n would increase at rate $\delta(1 - n)$. This dynamics implies that at p_3, the probability of the system bifurcating up is proportional to the relative rate at which it goes up: $\frac{\delta(1-n)}{\delta(1-n)+\delta(n-\alpha)} = \frac{1-n}{1-\alpha}$.

3.1. Vanishing shocks

Consider the limiting case in which shocks to fundamentals vanish, that is, $\mu \to 0$ and $\sigma \to 0$. Let $\overline{Z}(n)$ and $\underline{Z}(n)$ denote the two types' equilibrium thresholds. The equilibrium properties depend on the degree of payoff heterogeneity.

The relative position of the dominance regions for the two types of agents on the $\mathbb{R} \times [0, 1]$ space reflects the degree of heterogeneity in their payoff functions. For a sufficiently large degree of heterogeneity, we have that $\overline{P}(n) < \underline{O}(n)\ \forall n$: a high-type agent that holds the worst possible belief concerning future choices of others demands a smaller value of the fundamental to be indifferent between the two actions than a low-type agent under the most optimistic belief. This implies there is no region in the state space in which neither type has a dominant action. On the other hand, if heterogeneity is not too large, dominance regions can

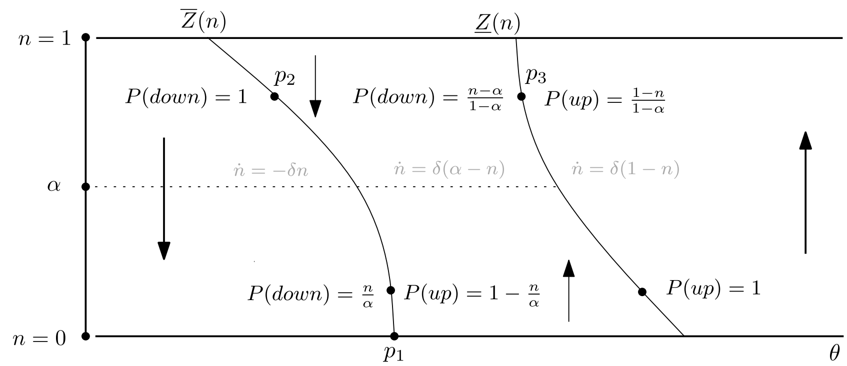

Fig. 3. Bifurcation probabilities.

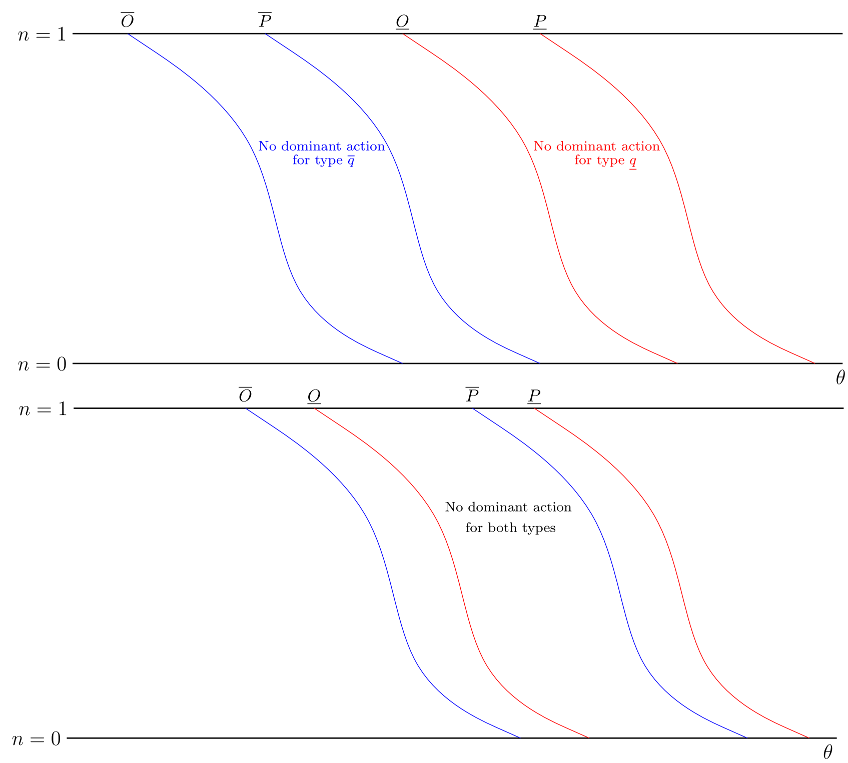

Fig. 4. Dominance regions — Large and small heterogeneity.

be such that $\underline{O}(n) < \overline{P}(n)\ \forall n$, so there is a region in which neither action is dominant for both types of agents. Fig. 4 exemplifies those two cases.

In the case of vanishing shocks, the upper dominance region boundary of a high-type agent can be computed as:

$$\int_0^\infty e^{-(\rho+\delta)t}\overline{\pi}(\overline{P}, n_t^\downarrow)dt = 0, \tag{2}$$

where $n_t^\downarrow = n_0 e^{-\delta t}$. The lower dominance region boundary of a low-type agent is given by

$$\int_0^\infty e^{-(\rho+\delta)t}\underline{\pi}(\underline{O}, n_t^\uparrow)dt = 0, \tag{3}$$

where $n_t^\uparrow = 1-(1-n_0)e^{-\delta t}$.

Expressions for the equilibrium thresholds are provided in Appendix A.1. Proposition 2 shows the main equilibrium properties for the case of vanishing shocks.

Proposition 2. *Suppose there are two types of agents in the economy, $\underline{q}$ and $\overline{q}$, with payoff functions given by $\underline{\pi}(\theta, n)$ and $\overline{\pi}(\theta, n)$, respectively, with $\overline{\pi}(.) > \underline{\pi}(.)\ \forall(\theta, n)$. In the limit as $\mu, \sigma \to 0$, in the unique rationalizable equilibrium:*

(i) if $\underline{O}(n) > \overline{P}(n)\ \forall n$, then $\overline{Z}(n) < \underline{Z}(n)\ \forall n$, so different types' thresholds do not intersect;

(ii) if $\underline{O}(n) < \overline{P}(n)\ \forall n$, then $\overline{Z}(n) = \underline{Z}(n)$ for all n in an interval containing α. Moreover, there are neighborhoods around 0 and 1 in which $\overline{Z}(n) < \underline{Z}(n)$.

Proof. See Appendix B.3. □

The first part of Proposition 2 states that when there is a lot of heterogeneity so that there is no intersection between the regions in which each type does not have a dominant strategy, each type of agent will play according to a distinct threshold. Their equilibrium thresholds will never coincide, which is not surprising since that would require some agents to play a strictly dominated strategy.

The second part of Proposition 2 brings a surprising result: there is some conformity in agents' behavior as long as heterogeneity is

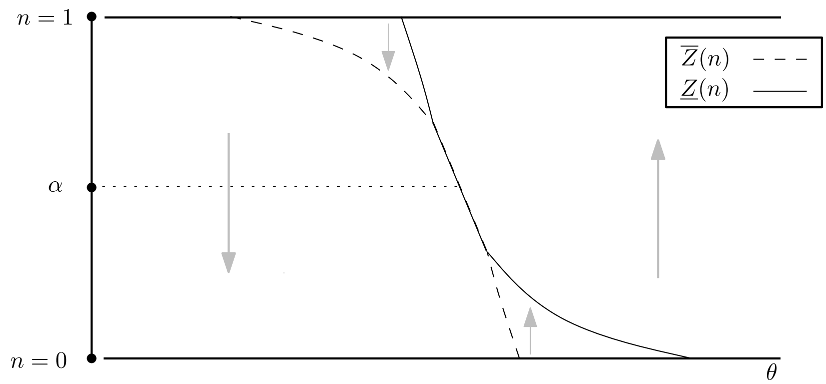

Fig. 5. Not so large heterogeneity.

not large enough to make it impossible for agents to play according to the same threshold for any (arbitrary) set of beliefs. Proposition 2 also states that different players will choose according to the same threshold for an intermediate range of n. Their thresholds will never fully coincide though: for extreme values of n, heterogeneity beats coordination and each type has a distinct threshold. Fig. 5 exemplifies this result.

In order to understand the result in Proposition 2, suppose agents play according to different thresholds as in Fig. 3. For $n = \alpha$, an agent at the lower threshold (the one at the left) holds the most pessimistic beliefs, n will surely decrease from then on. That is because all low-type agents will be choosing 0, and at $n = \alpha$ they are just enough to determine the path of the economy. Hence an agent will not choose action 1 unless it is dominant to do so. Conversely, an agent at the higher threshold (the one at the right) holds the most optimistic beliefs for exactly the same reason: high-type agents are choosing 1 and at $n = \alpha$ they are just enough to drive the economy up. Hence an agent will not choose 0 unless it is dominant to do so.

This reasoning implies that an equilibrium with two distinct thresholds at $n = \alpha$ requires (i) high-type agents being indifferent between either choice for some $\tilde{\theta}$ holding the most pessimistic beliefs; and (ii) low-type agents being indifferent between either choice for some $\theta > \tilde{\theta}$ holding the most optimistic beliefs. This can only happen in case of very large payoff heterogeneity. If that is not the case, owing to the large dispersion in beliefs offsetting idiosyncratic payoffs, both thresholds will coincide at $n = \alpha$.

This reasoning also explains why conformity fails to arise for extreme values of n. As shown in Fig. 5, when $n = 0$, beliefs at both thresholds are not so different: both types know the economy will move up. The speed is not the same in both cases, but that is a minor difference in beliefs. Hence even a small difference in preferences leads to the existence of two distinct thresholds.

In sum, for intermediate values of n, there is huge heterogeneity in expectations about the path of n around the equilibrium threshold, which makes payoff heterogeneity less relevant. In contrast, for extreme values of n, there is less uncertainty about the path of n around the equilibrium thresholds and hence heterogeneity in preference matters for agents' optimal choice.

3.2. Vanishing frictions

We now consider the limiting case of $\delta \to \infty$ so that agents receive very frequent opportunities to revise their actions. Expressions for equilibrium thresholds under a general payoff function and vanishing frictions are provided in Appendix A.2. Proposition 3 emphasizes some properties of the equilibrium when distinct types' flow payoffs differ by a constant.

Proposition 3. *Let $\overline{\pi}(\theta, n) = \pi(\theta, n) + \overline{\varepsilon}$ and $\underline{\pi}(\theta, n) = \pi(\theta, n) + \underline{\varepsilon}$, where $\pi(.)$ is continuously differentiable and strictly increasing in both arguments and $\overline{\varepsilon} > \underline{\varepsilon}$. Define $\hat{\varepsilon} \equiv \alpha\overline{\varepsilon} + (1-\alpha)\underline{\varepsilon}$ and $\overline{z}^*$, $\underline{z}^*$ and $\hat{z}^*$ as satisfying $\int_0^\alpha \pi(\overline{z}^*, n)dn = -\alpha\overline{\varepsilon}$, $\int_\alpha^1 \pi(\underline{z}^*, n)dn = -(1-\alpha)\underline{\varepsilon}$ and $\int_0^1 \pi\left(\hat{z}^*, n\right) dn = -\hat{\varepsilon}$, respectively. In the limit as $\delta \to \infty$:*

(i) if $\underline{O}(n) > \overline{P}(n)$ $\forall n$, the state space is divided in three regions: whenever $\theta_t > \underline{z}^*$, $n_t \approx 1$; whenever $\overline{z}^* < \theta_t < \underline{z}^*$, $n_t \approx \alpha$ and whenever $\theta_t < \overline{z}^*$, $n_t \approx 0$.

(ii) if $\underline{O}(n) \leq \overline{P}(n)$ $\forall n$, the vertical line $\hat{z}^*$ divides the state space in two regions: whenever $\theta_t > \hat{z}^*$, $n_t \approx 1$ and whenever $\theta_t < \hat{z}^*$, $n_t \approx 0$.

Proof. See Appendix B.4. □

Proposition 3 states that in case of very large heterogeneity, at a given point in time, (almost) all agents of a given type will be playing the same action but different types might be playing different actions. The bounds of the region where behavior is heterogeneous (the switching point for each group) are determined by the value of θ such that, at $n = \alpha$: (i) high types with pessimistic beliefs are indifferent between either action; and (ii) low types with optimistic beliefs are indifferent between either action.

When heterogeneity is not so large, in the limiting case of vanishing frictions, the economy behaves as if agents were identical and had an intermediate preference parameter $\hat{\varepsilon}$. Although agents' strategies differ, whenever fundamentals cross the vertical division line, all agents of a certain type immediately switch actions, leading the opposite type to consider it profitable to switch actions as well. Since chances to switch arrive at a very large rate, the dynamics of the economy is basically the same as if agents were identical with preferences given by $\hat{\pi}(\theta, n) = \pi(\theta, n) + \hat{\varepsilon}$. Hence the economy behaves as in Frankel and Pauzner (2000). In the limiting case of vanishing frictions, two networks can only coexist if there is no set of (arbitrary) beliefs that would lead different agents to play according to the same threshold.

Fig. 6 depicts the equilibrium in case of not so large heterogeneity. Note that agents' strategies differ for values of n close to 0 and 1. As explained before, that is because for high and low values of n, beliefs at both thresholds are not so different, so payoff heterogeneity matters. This intuition is not affected when δ is large – agents at $n = 0$ at the right of $\hat{z}^*$ know n will be moving up fast, but also that they will quickly get another chance to choose.

3.2.1. The case of $\delta \to \infty$ as an equilibrium selection device

In the case of vanishing frictions, for almost all values of θ, n is almost constant. Hence this case can be seen as an equilibrium selection device. We can thus compare the results from Proposition 3 with those from static coordination games.

Consider a two-action static game analogous to the one we study, i.e., the relative payoff from action 1 is given by $\pi_{q(i)}(\theta, n)$.

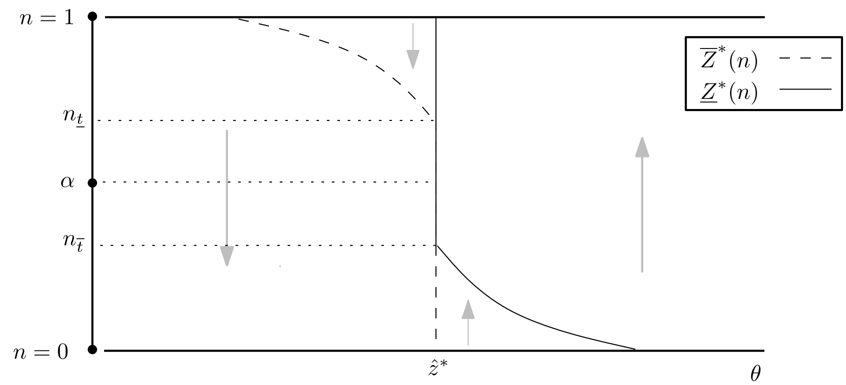

Fig. 6. Vanishing timing frictions.

A potential function $\mathcal{P}$ is defined as follows: suppose that an infinitesimal measure dn of type-q agents switches behavior from 0 to 1; $\mathcal{P}$ is a potential function if this raises $\mathcal{P}$ by $\pi_q(\theta, n)\, dn$.[10]

Under the preferences described in Proposition 3, this two-action static game is a potential game with potential function given by[11]

$$\mathcal{P}\left(\overline{n}, \underline{n}; \theta\right) = \int_0^{\alpha\overline{n}+(1-\alpha)\underline{n}} \pi(\theta, u)du + \alpha\overline{\varepsilon}\overline{n} + (1-\alpha)\underline{\varepsilon}\underline{n}. \tag{4}$$

Proposition 4 shows that the result from the maximization of the potential in (4) coincides with the outcome from Proposition 3.

Proposition 4. *Consider $\overline{\pi}(\theta, n)$, $\underline{\pi}(\theta, n)$, $\overline{z}^*$, $\underline{z}^*$ and $\hat{z}^*$ as in Proposition 3. Maximization of (4) implies that*

(i) *if $\underline{O}(n) > \overline{P}(n)\ \forall n$, the potential maximizer is given by*

$$\arg\max \mathcal{P}(\overline{n}, \underline{n}; \theta) = \begin{cases} (0,0) & \text{for } \theta < \overline{z}^*, \\ \{(0,0),(1,0)\} & \text{for } \theta = \overline{z}^*, \\ (1,0) & \text{for } \theta \in (\overline{z}^*, \underline{z}^*), \\ \{(1,0),(1,1)\} & \text{for } \theta = \underline{z}^*, \\ (1,1) & \text{for } \theta > \underline{z}^*; \end{cases}$$

(ii) *if $\underline{O}(n) < \overline{P}(n)\ \forall n$, the potential maximizer is given by*[12]

$$\arg\max \mathcal{P}\left(\overline{n}, \underline{n}; \theta\right) = \begin{cases} (0,0) & \text{for } \theta < \hat{z}^*, \\ \{(0,0),(1,1)\} & \text{for } \theta = \hat{z}^*, \\ (1,1) & \text{for } \theta > \hat{z}^*. \end{cases}$$

Proof. See Appendix B.5. □

Proposition 3 characterizes the outcome of a dynamic game. Proposition 4 shows that the same result is obtained by considering the potential maximizer of the analogous static game.

The literature on global games provides an alternative way of studying the outcome of coordination games. Embedding the static game considered in this section into a global game means that agents have access to noisy idiosyncratic information about θ and simultaneously choose between actions 0 and 1. The results in Frankel et al. (2003) imply that, as noise becomes arbitrarily small, the outcome of the global game coincides with the potential maximizer (irrespective of the noise structure). Hence it also coincides with the equilibrium selected in the dynamic game with vanishing frictions.

10 See Monderer and Shapley (1996) and Sandholm (2001).

11 Given anonymity within each type, actions can be summarized by a pair $(\overline{n}, \underline{n})$. We omit the parametrization by θ whenever this does not cause confusion.

12 In the case where $\overline{P}(n) = \underline{O}(n)\ \forall n$, the set of potential maximizers is essentially the same as when $\underline{O}(n) < \overline{P}(n)\ \forall n$, except for the fact that $\arg\max \mathcal{P}(\overline{n}, \underline{n}; \hat{z}^*) = \{(1,1),(0,0),(1,0)\}$, as shown in Appendix B.5.

4. Linear payoff function

We now present a linear example that provides intuition on the forces at play and helps us to understand the relative effects of payoff heterogeneity and complementarities in preferences. Let the relative flow-payoff of action 1 in comparison to action 0 be given by:

$$\pi_i(\theta_t, n_t) = \theta_t + \gamma n_t + \varepsilon_i$$

with

$$\varepsilon_i = \begin{cases} \overline{\varepsilon} & \forall i \in [0, \alpha] \\ \underline{\varepsilon} & \forall i \in (\alpha, 1], \end{cases}$$

that is, there are two types of agents: a proportion α with preference parameter $\overline{\varepsilon}$ and a proportion $1-\alpha$ with preference parameter $\underline{\varepsilon}$, $\overline{\varepsilon} > \underline{\varepsilon}$.

As before, we will analyze the two limiting cases in which we can derive analytical results, starting by the case of slow-moving fundamentals.

4.1. Vanishing shocks

Consider again the case of $\mu, \sigma \to 0$. First, we compute the two dominance regions' boundaries that can be used to measure the degree of heterogeneity in agents' payoffs. Substituting our linear payoff function in Eqs. (2) and (3) and solving the integrals yields the upper dominance region boundary of a high-type agent:

$$\overline{P}(n_0) = -\overline{\varepsilon} - \frac{\gamma(\rho+\delta)}{\rho+2\delta} n_0, \tag{5}$$

and the lower dominance region boundary of a low-type agent:

$$\underline{O}(n_0) = -\underline{\varepsilon} - \frac{\gamma\delta}{\rho+2\delta} - \frac{\gamma(\rho+\delta)}{\rho+2\delta} n_0. \tag{6}$$

Large heterogeneity

The condition ensuring that $\overline{P}(n_0) < \underline{O}(n_0)\ \forall n_0$ is equivalent to

$$\overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma\delta}{\rho+2\delta}. \tag{7}$$

If the difference between idiosyncratic preference parameters is large enough in comparison to the importance of strategic complementarities (γ), the curve on which a high-type agent with pessimistic beliefs about n is indifferent between 0 and 1 is located to the left of the curve on which a low-type agent with optimistic beliefs is indifferent between the two actions. The intersection between the region in which neither action is dominant for a high-type agent and the region with no dominant action for a low-type agent is empty (as in the top picture of Fig. 4).

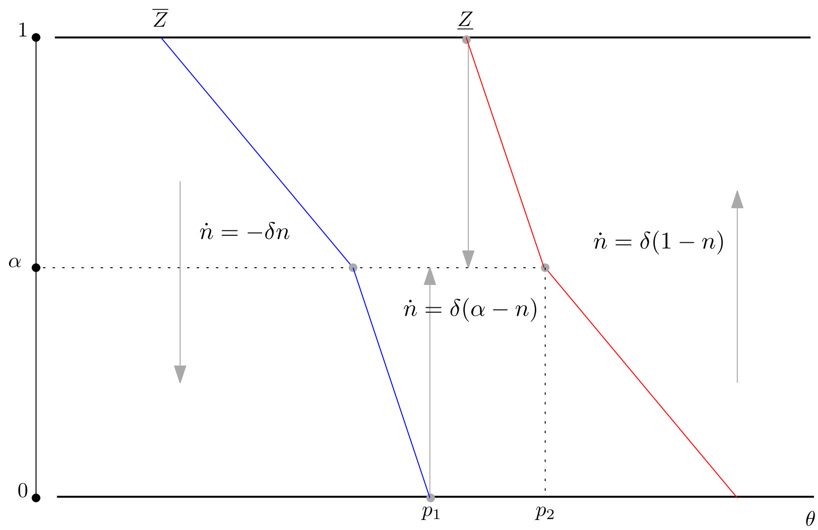

Fig. 7. Case $\overline{\varepsilon} - \underline{\varepsilon} \geq \frac{\gamma(\delta+\rho\max\{\alpha,1-\alpha\})}{\rho+2\delta}$.

If condition (7) holds, then there is no set of beliefs that could induce different agents to play according to the same strategy for any value of n_0. Hence, whenever this condition is satisfied, the equilibrium in the limit as $\mu, \sigma \to 0$ will be such that type-$\overline{\varepsilon}$ and type-$\underline{\varepsilon}$ agents play according to thresholds that do not intersect, as stated in Proposition 2.

How can we analytically compute the thresholds in this case?[13] First note that for all $n \geq \alpha$ the high-type threshold coincides with the high-type upper dominance region. The belief a type-$\overline{\varepsilon}$ agent holds in equilibrium at some point (θ, n) with $n \geq \alpha$ is that n will fall at the maximum rate with probability one (see Fig. 3). Under the most pessimistic belief, this agent is indifferent between the two actions. Thus, type-$\overline{\varepsilon}$ agents' threshold above α is a function $\overline{Z}(n_0) = \overline{P}(n_0)$, and thus satisfies Eq. (5). Likewise, for all $n \leq \alpha$ the low-type threshold must coincide with the low-type lower dominance region (in which agents hold the most optimistic belief), so for $n_0 \leq \alpha$, $\underline{O}(n_0)$ is given by Eq. (6).

What about the high-type threshold below α and the low-type threshold above α? Suppose, for now, that heterogeneity is large enough so that the following inequality holds:

$$\overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma(\delta + \rho\alpha)}{\rho + 2\delta}. \tag{8}$$

This condition ensures the equilibrium is such that $\overline{Z}(0) < \underline{Z}(\alpha)$, so that if the economy is initially at some point on $\overline{Z}(n_0)$ with $n_0 < \alpha$, it will never reach $n = 1$: it will either go down towards $n = 0$ or up towards $n = \alpha$. In other words, the system will never cross the other type's threshold. Graphically, it means that p_1 is to the left of p_2 in Fig. 7. We can then compute the high-type equilibrium threshold below α as follows:

$$\underbrace{\frac{(\alpha - n_0)}{\alpha}}_{P(\text{up})} \int_0^\infty e^{-(\rho+\delta)t} \overline{\pi}(\overline{Z}, \underbrace{\alpha - (\alpha - n_0)e^{-\delta t}}_{n_t \text{ growing towards } \alpha})dt + \underbrace{\frac{n_0}{\alpha}}_{P(\text{down})} \int_0^\infty e^{-(\rho+\delta)t} \overline{\pi}(\overline{Z}, \underbrace{n_0 e^{-\delta t}}_{n_t \text{ falling}})dt = 0.$$

The first term of the sum is the probability of an upward bifurcation times the discounted relative payoff of action 1 when the agent expects n_t to grow until it approaches α. The second term is the probability of a downward bifurcation times the discounted payoff when the agent expects n_t to decrease towards zero. Substituting our linear functional form for $\overline{\pi}(.)$ and solving the integrals, we have that, whenever (8) holds, $\overline{Z}$ is given by

$$\overline{Z}(n_0) = \begin{cases} -\overline{\varepsilon} - \dfrac{\gamma(\rho+\delta)}{\rho+2\delta} n_0 & \text{if } n_0 \geq \alpha \\ -\overline{\varepsilon} - \dfrac{\alpha\gamma\delta}{\rho+2\delta} - \dfrac{\gamma\rho}{\rho+2\delta} n_0 & \text{if } n_0 < \alpha. \end{cases} \tag{9}$$

Analogous expressions for the low-type equilibrium threshold are derived in Appendix A.3. Fig. 7 depicts the equilibrium in case $\overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma(\delta+\rho\max\{\alpha,1-\alpha\})}{\rho+2\delta}$, that is, the case in which if the economy starts at one threshold, it will never cross the other one.[14]

Now, suppose heterogeneity is still large so that $\overline{P}$ is to the left of $\underline{O}$, but not as large as before. Specifically, assume

$$\frac{\gamma\delta}{\rho+2\delta} < \overline{\varepsilon} - \underline{\varepsilon} \leq \frac{\gamma(\delta+\rho\alpha)}{\rho+2\delta}. \tag{10}$$

Define $n' \equiv \alpha - \frac{(\rho+2\delta)(\overline{\varepsilon}-\underline{\varepsilon})-\gamma\delta}{\gamma\rho}$.[15] The threshold $\overline{Z}$ is still given by Eq. (9) for all $n_0 \geq n'$, but for all $n_0 < n'$, it satisfies:

$$\underbrace{\frac{(\alpha - n_0)}{\alpha}}_{P(\text{up})} \left\{ \int_0^{\bar{t}} e^{-(\rho+\delta)t} \overline{\pi}(\overline{Z}, \underbrace{\alpha - (\alpha - n_0)e^{-\delta t}}_{n_t \text{ growing towards } \alpha})dt + \int_{\bar{t}}^\infty e^{-(\rho+\delta)t} \overline{\pi}(\overline{Z}, \underbrace{1 - (1 - n_{\bar{t}})e^{-\delta(t-\bar{t})}}_{n_t \text{ growing towards } 1})dt \right\} \underbrace{\frac{n_0}{\alpha}}_{P(\text{down})} \int_0^\infty e^{-(\rho+\delta)t} \overline{\pi}(\overline{Z}, \underbrace{n_0 e^{-\delta t}}_{n_t \text{ falling}})dt = 0 \tag{11}$$

where $\bar{t}$ is the time at which the system reaches the other type's threshold, which is given by $\bar{t} = -\frac{1}{\delta}\ln\frac{\alpha - n_{\bar{t}}}{\alpha - n_0}$, and $n_{\bar{t}} = \underline{Z}^{-1}\left(\overline{Z}(n_0)\right) = -\left[(\rho+2\delta)\left(\overline{Z} + \underline{\varepsilon}\right) + \gamma\delta\right]/[(\rho+\delta)\gamma]$.

[13] All the following derivations are equivalent to directly applying Proposition 5 in Appendix A.1 for the case of linear payoffs.

[14] This condition is analogous to $\overline{Z}_0 < \underline{Z}_\alpha$ and $\overline{Z}_\alpha < \underline{Z}_1$ in Proposition 5.

[15] The bound n' is the value satisfying $\overline{Z}(n') = \underline{Z}(\alpha)$.

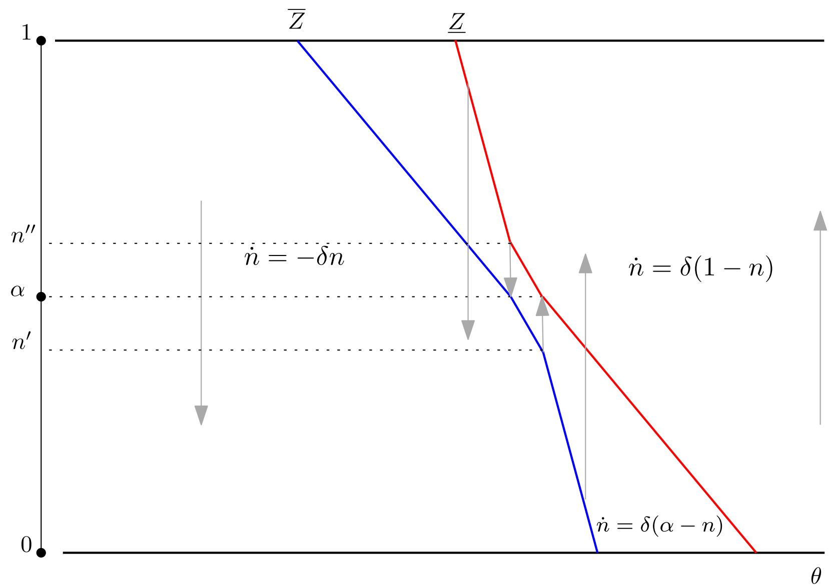

Fig. 8. Case $\frac{\gamma\delta}{\rho+2\delta} < \overline{\varepsilon} - \underline{\varepsilon} \le \frac{\gamma\delta+\gamma\rho\min\{\alpha,1-\alpha\}}{\rho+2\delta}$.

The expression in (11) can be better understood with the aid of Fig. 8.[16] There is a range of n (n is sufficient low) such that a type-$\overline{\varepsilon}$ agent on her threshold knows that, if the system bifurcates up, it will cross the other type's threshold at some point, and thereafter n will grow at a higher rate. Then, given this more optimistic belief, the increase in the level of fundamentals an agent demands to be indifferent between the two actions for a given decrease in n_0 is smaller, i.e., the threshold is steeper.

Using the fact that $\underline{Z}(n) = \underline{O}(n)$ for all $n \le \alpha$ and performing a change of variables in Eq. (11), we find that, whenever (10) holds, $\forall n_0 < n'$, $\overline{Z}$ satisfies

$$(\rho+2\delta)(\overline{Z}+\overline{\varepsilon}) + \gamma\rho n_0 + \gamma\delta\alpha + \gamma\delta\frac{(1-\alpha)}{\alpha}\left(\frac{1}{\alpha-n_0}\right)^{\frac{\rho}{\delta}}\left[\alpha + \frac{(\overline{Z}+\underline{\varepsilon})(\rho+2\delta)+\gamma\delta}{\gamma(\rho+\delta)}\right]^{\frac{\rho+\delta}{\delta}} = 0. \quad (12)$$

Not so large heterogeneity

Finally, consider the case in which $\overline{P}(n_0) \ge \underline{O}(n_0)$ $\forall n_0$, which is equivalent to

$$\overline{\varepsilon} - \underline{\varepsilon} \le \frac{\gamma\delta}{\rho+2\delta}. \quad (13)$$

We know by Proposition 2 that different-type agents' strategies will coincide for some values of n, but never fully coincide. The equilibrium in this case is as depicted in Fig. 9.

For all $n_0 \le \hat{n}$, the type-$\overline{\varepsilon}$ threshold is identical to $\overline{P}(n_0)$, and for all $n_0 \ge \hat{\hat{n}}$, the type-$\overline{\varepsilon}$ threshold is given by (12). Analogous equations describing the low-type threshold are provided in Appendix A.3. For $n_0 \in [\hat{n}, \hat{\hat{n}}]$, the two types act according to a single threshold, given by

$$Z(n_0) = -\hat{\varepsilon} - \frac{\gamma\delta}{\rho+2\delta} - \frac{\gamma\rho n_0}{\rho+2\delta},$$

which is exactly what the threshold would be like if agents were all identical to an intermediate type with preference parameter $\hat{\varepsilon} = \alpha\overline{\varepsilon} + (1-\alpha)\underline{\varepsilon}$.[17]

In equilibrium, there is conformity in agents' strategies for intermediate values of n as long as the condition in (13) holds. In most applications, ρ (time discount rate) is much smaller than δ (frequency of opportunities to revise behavior), so the expression in (13) can be approximated by $\overline{\varepsilon} - \underline{\varepsilon} \le \gamma/2$. In words, an equilibrium where agents play according to different thresholds requires payoff heterogeneity to be as important as an increase in n equal to half of the population. When (13) does not hold, there is no set of beliefs that would make conformity possible as $\overline{P}$ is to the left of $\underline{O}$.

The bounds $\hat{n}$ and $\hat{\hat{n}}$ in Fig. 9 are given by

$$\hat{n} = \alpha\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)}{\gamma\delta},$$

$$\hat{\hat{n}} = 1-(1-\alpha)\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)}{\gamma\delta}.$$

In case $\alpha = 1/2$ and ρ is much smaller than δ, we get $\hat{n} \approx (\overline{\varepsilon}-\underline{\varepsilon})/\gamma$, thus $\hat{n}$ is approximately equal to the increase in n that would compensate the idiosyncratic difference in payoffs.

Intuitively, the existence of type-$\overline{\varepsilon}$ agents increases incentives for type-$\underline{\varepsilon}$ to choose action 1, while the existence of type-$\underline{\varepsilon}$ agents increases incentives for type-$\overline{\varepsilon}$ to opt for 0. In consequence, agents behave in a more similar way. That is particularly true when n is in an intermediate range so that the path of the economy will be decided by the actions of both groups.

4.2. Vanishing frictions

Our linear payoff function satisfies the assumptions in Proposition 3. In order to compute the equilibrium in the limit case of vanishing timing frictions (even if $\mu, \sigma > 0$), it suffices to apply the results in Proposition 6, or equivalently, take the limit as $\delta \to \infty$ of all equilibrium thresholds computed in the previous subsection.

[16] The second integral in Eq. (11) is equivalent to $\int_t^\infty e^{-(\rho+\delta)t}(\overline{\varepsilon}-\underline{\varepsilon})dt$, because when the system arrives at $\underline{Z}$ and is expected to grow towards 1, low-type agents have zero payoff. A high-type agent holding the same belief will have a payoff larger by $\overline{\varepsilon}-\underline{\varepsilon}$ at all future dates. This substitution facilitates the algebra considerably.

[17] We cannot apply the bifurcation probabilities to compute the thresholds when they coincide. However, it is possible recover the coinciding part of the thresholds from the planner's solution – under different payoffs. For details, see Appendix A.3.

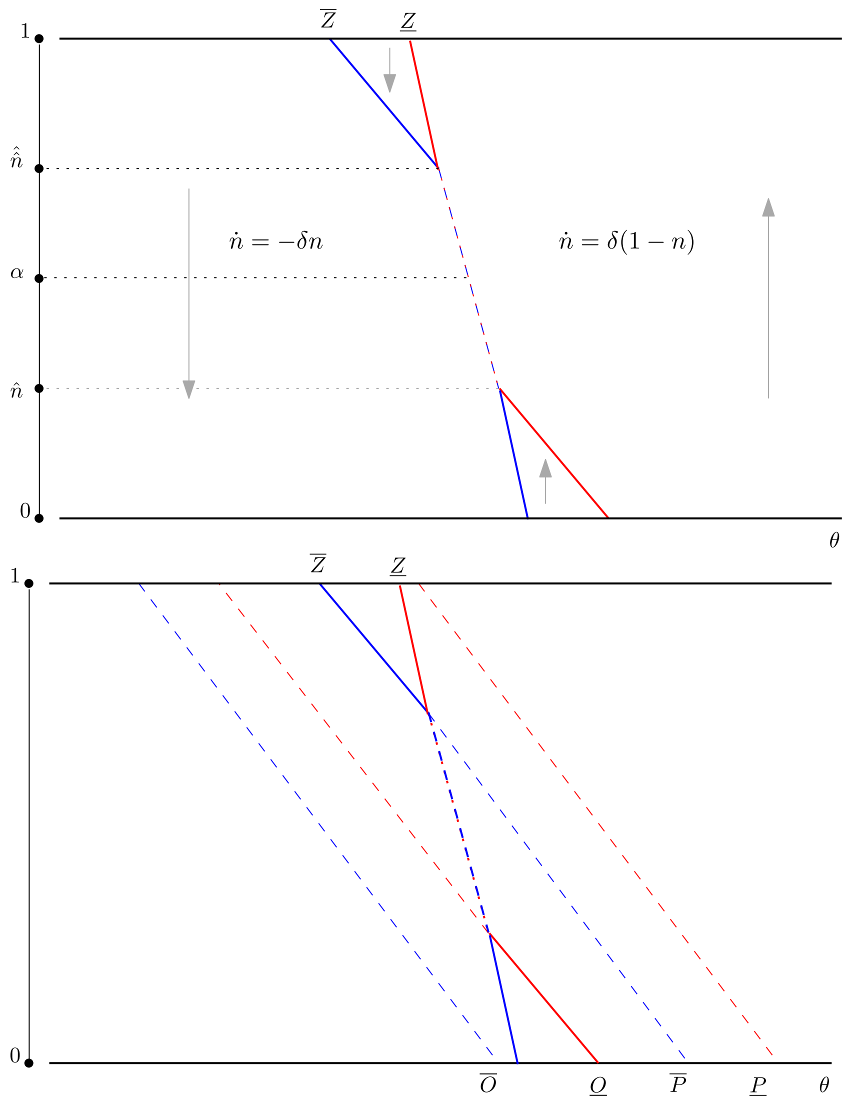

Fig. 9. Case $\overline{\varepsilon} - \underline{\varepsilon} \le \frac{\gamma\delta}{\rho+2\delta}$.

The more interesting case is when $\underline{O}$ is to the left of $\overline{P}$, which is equivalent to $\overline{\varepsilon} - \underline{\varepsilon} \le \gamma/2$. Equilibrium is as in Fig. 10. The line $\hat{Z}$ divides the state space in two regions: whenever $\theta_t > \hat{Z}, n_t \approx 1$, and whenever $\theta_t < \hat{Z}, n_t \approx 0$. $\hat{Z}$ coincides with the equilibrium that would be played if there was just one type of agent in the economy with preference parameter $\hat{\varepsilon} \equiv \alpha\overline{\varepsilon} + (1-\alpha)\underline{\varepsilon}$. $\hat{Z}$ satisfies

$$\hat{Z} = -\hat{\varepsilon} - \gamma/2.$$

The increase in n when θ crosses to the right of $\hat{Z}$ at $n \approx 0$ is triggered by high-type agents choosing action 1, as low-type agents initially keep choosing 0. However, this difference in behavior will be very short lived. One implication of this result is that an increase in the mass of high-type agents (α) will affect the behavior of everyone in the economy (it will shift the threshold to the left) but there will be virtually no difference in the behavior of low-type and high-type agents.

5. The planner's problem

We now solve the planner's problem for the case of linear payoffs in order to analyze efficiency in this environment. All results in this section refer to the case of very small shocks, $\mu, \sigma \to 0$.

In order to solve the planner's problem, we need to specify the flow utilities of each option, and not just the difference in payoffs. Consider that the flow utility agent i derives from being committed to action 1 is given by $u_i^1(\theta_t^1, n_t) = \theta_t^1 + \nu^1 n_t + \varepsilon_i^1$, and the flow utility from being at 0 is given by $u_i^0(\theta_t^0, n_t) = \theta_t^0 + \nu^0(1-n_t) + \varepsilon_i^0$. n_t is the mass of agents currently playing 1, $\nu^j > 0$ is a parameter measuring the relative importance of strategic complementarities in the choice of j, θ_t^j represents the fundamentals affecting the flow-payoff of playing j at time t, and ε_i^j captures an idiosyncratic preference for action j, $j \in \{0, 1\}$. θ_t^j follows a Brownian motion with drift μ_j and variance σ_j^2. Defining $\theta_t \equiv \theta_t^1 - \theta_t^0 - \nu^0$, $\gamma \equiv \nu^1 + \nu^0$ and $\varepsilon_i \equiv \varepsilon_i^1 - \varepsilon_i^0$, we can write the relative payoff function as before: $\pi(\theta_t, n_t) = \theta_t + \gamma n_t + \varepsilon_i$. θ_t follows a Brownian motion with drift $\mu = \mu^1 - \mu^0$ and variance $\sigma^2 = \sigma_0^2 + \sigma_1^2$.

We will refer to options 0 and 1 as networks, since the measure of agents playing each action generates externalities that can be thought of as network effects.

Consider the case $\nu^0 = \nu^1 = \nu$. At every point in time, the planner chooses the proportion of high- and low-type agents with an opportunity to revise their actions that will opt for network 1 in order to maximize aggregate welfare. Denote by $\overline{\phi}_t$ and $\underline{\phi}_t$ these proportions, respectively. At time 0, the planner maximizes the

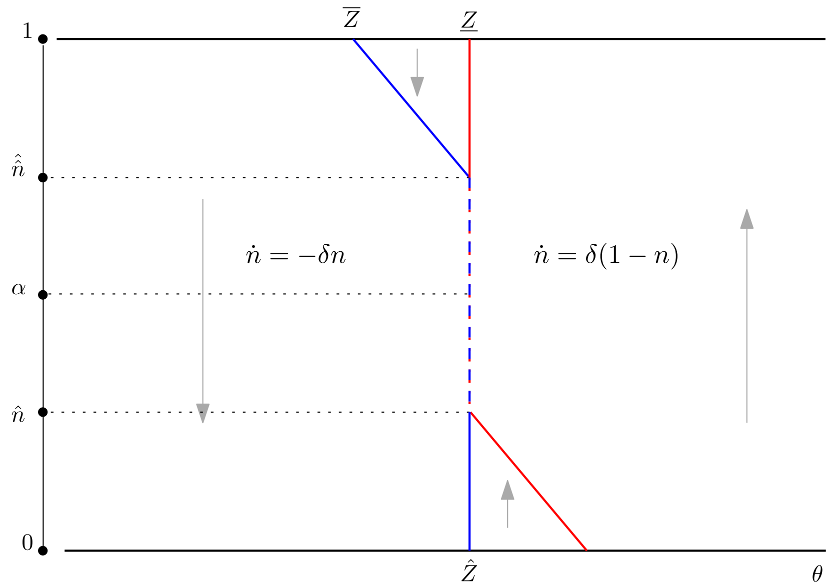

Fig. 10. Small heterogeneity and vanishing frictions.

discounted sum of utilities across agents, i.e.,

$$\mathbb{E}\,\alpha\int_0^\infty e^{-\rho t}\left\{\overline{n_t}\left[{\theta_t}^1+\nu n_t+\overline{\varepsilon}^1\right]\right.$$
$$\left.+\,(1-\overline{n_t})\left[{\theta_t}^0+\nu(1-n_t)+\overline{\varepsilon}^0\right]\right\}dt$$
$$+(1-\alpha)\int_0^\infty e^{-\rho t}\left\{\underline{n}_t\left[{\theta_t}^1+\nu n_t+\underline{\varepsilon}^1\right]\right.$$
$$\left.+\,(1-\underline{n}_t)\left[{\theta_t}^0+\nu(1-n_t)+\underline{\varepsilon}^0\right]\right\}dt,$$

which is equivalent to maximizing

$$\mathbb{E}\int_0^\infty e^{-\rho t}\left\{n_t\left[\theta_t-\frac{\gamma}{2}+\gamma n_t\right]+\alpha\overline{n_t\varepsilon}+(1-\alpha)\underline{n_t\varepsilon}\right\}dt, \qquad (14)$$

where, by definition, $n_t=\alpha\overline{n}_t+(1-\alpha)\underline{n}_t$.

Suppose the planner's optimal choice at $t=0$ is $\left\{\overline{\phi}_0,\underline{\phi}_0\right\}$, with $\overline{\phi}_0\in[0,1)$. Consider a deviation in which the planner chooses $\overline{\phi}_0=1$ and keeps $\underline{\phi}_0$ and all future choices of $\overline{\phi}_t$ and $\underline{\phi}_t$ unchanged, for any realization of the Brownian path. Such deviation implies an infinitesimal increase in $\overline{n}_0$ by $d\overline{n}_0$, and its effect on future values of $\overline{n}_t$ is given by $d\overline{n}_t=d\overline{n}_0e^{-\delta t}$, since the initial increase in $\overline{n}_0$ depreciates at a rate δ. A necessary condition for optimality of the planner's choice is that the deviation just described is not profitable. Using (14), it means that if $\overline{\phi}_0\in[0,1)$ is optimal, it cannot be the case that

$$\mathbb{E}\int_0^\infty\frac{\partial e^{-\rho t}\left\{n_t\left[\theta_t-\frac{\gamma}{2}+\gamma n_t\right]+\alpha\overline{n_t\varepsilon}+(1-\alpha)\underline{n_t\varepsilon}\right\}}{\partial\overline{n}_t}\frac{d\overline{n}_t}{d\overline{n}_0}dt>0,$$

which can be written as

$$\mathbb{E}\int_0^\infty e^{-(\rho+\delta)t}\left(\theta_t-\frac{\gamma}{2}+2\gamma n_t+\overline{\varepsilon}\right)dt>0. \qquad (15)$$

Hence, if the condition in (15) holds, it must be optimal for the planner to set $\overline{\phi}_0=1$, that is, the planner recommends action 1 for all high-type agents with a revision opportunity in hand. If the inequality in (15) is reversed, action 0 is optimal for high-type agents at $t=0$. An analogous reasoning implies that whenever

$$\mathbb{E}\int_0^\infty e^{-(\rho+\delta)t}\left(\theta_t-\frac{\gamma}{2}+2\gamma n_t+\underline{\varepsilon}\right)dt>0, \qquad (16)$$

action 1 is optimal for low-type agents, and if the inequality is reversed, action 0 is optimal for low-type agents, at $t=0$. These expressions are similar to the indifference condition in the decentralized equilibrium: an agent is indifferent between 0 and 1 when $\mathbb{E}\int_{t=0}^\infty e^{(\rho+\delta)t}\left[\theta+\gamma n_t+\varepsilon_i\right]dt=0$, $\varepsilon_i\in\left\{\overline{\varepsilon},\underline{\varepsilon}\right\}$. One important difference is that the externality is more important for the planner (γ is multiplied by 2). Intuitively, the planner takes into account the externality on others that agents fail to internalize, while the intrinsic quality of each option and idiosyncratic tastes are fully taken into account by agents in the decentralized equilibrium.

Mathematically, the planner's problem and the one solved by agents in the decentralized equilibrium are very similar: at every time t, the planner chooses according to the conditions in (15) and (16), knowing that its future selves will act optimally at all future dates, so the path of n must be consistent with optimality.[18] Hence the planner's problem is isomorphic to a game played by agents, only with different payoffs.

Proposition 7 in Appendix A.4 characterizes the planner's solution analytically. The main result in this section is that the region in which the planner prescribes the same strategy for different types is always larger than in the decentralized equilibrium. Hence, from the planner's point of view, there is not enough conformity. That is because the planner internalizes the externalities from agents' choices and thus puts a higher weight on coordination. A similar reasoning implies that in equilibrium, networks are 'too balanced' in the static model of Argenziano (2008).

Moreover, the planner's threshold is always flatter, meaning that the planner sacrifices gains stemming from good fundamentals in order to explore strategic complementarities. This effect is unrelated to heterogeneity and is analyzed in Guimaraes and Pereira (2016). Intuitively, agents do not internalize the effects of their choices on others' payoffs, hence they give relatively more importance to the intrinsic quality of each option.

Fig. 11 considers a case with large heterogeneity. In the decentralized equilibrium, for some values of θ, two networks will coexist for long periods of time, with some agents choosing 1 and others going for 0. However, the efficient outcome would feature a single network (except for brief transition periods). The strategies prescribed by the planner imply that n would almost always be very close to 0 or 1.

18 There are no commitment concerns here since the planner dictates everyone's actions and preferences exhibit no time inconsistency.

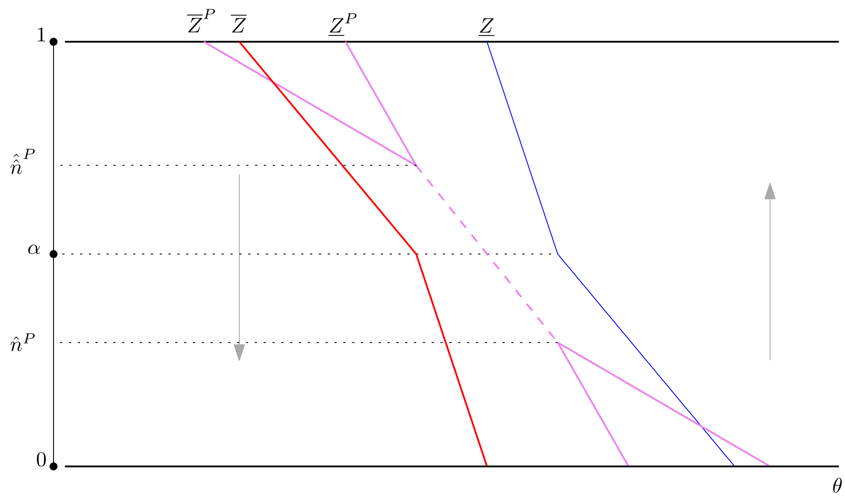

Fig. 11. Planner's solution when $\frac{\gamma(\delta+\rho\max\{\alpha,(1-\alpha)\})}{\rho+2\delta} < \overline{\varepsilon} - \underline{\varepsilon} \leq \frac{2\gamma\delta}{\rho+2\delta}$.

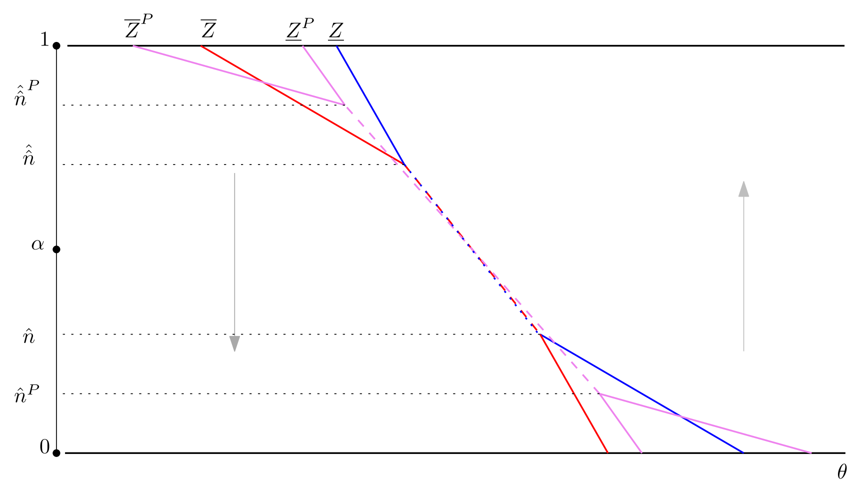

Fig. 12. Planner's solution when $\overline{\varepsilon} - \underline{\varepsilon} \leq \frac{\gamma\delta}{\rho+2\delta}$.

In the example in Fig. 12, heterogeneity is not so large and the equilibrium threshold of both types coincide for some values of n. In this case, the range of values of n for which agents choose different actions is (exactly) twice as large as the analogous range for the planner.

So far, we have focused on the case $\nu^0 = \nu^1$. However, the main results of this section hold when the network effect is asymmetric, that is, $\nu^0 \neq \nu^1$. In this case, the planner also shifts the threshold in order to enlarge the region in which agents choose the action that generates more externalities. But a parallel shift of the threshold is the only difference.[19]

6. Final remarks

This paper shows that in a dynamic coordination model with timing frictions, heterogeneous agents will often play similar strategies. Agents predisposed to a certain action will be less willing to take it anticipating the behavior of agents less inclined to choose that action, and vice versa. That is particularly true when there is an intermediate number of people in a network, in which case there is more uncertainty about the path of the economy and coordination motives dominate idiosyncratic tastes.

Network effects are key for internet companies, for example. Therefore, according to this paper, we should expect a lot of conformity in people's choices, hence a lot of concentration, but occasional large (positive or negative) shifts in the market share of these firms – which seems consistent with the stylized facts. Future research might build on this model to quantitatively analyze this kind of dynamic process.

Appendix A. Expressions for the equilibrium thresholds

A.1. Vanishing shocks

Characterization of equilibrium in the limiting case of vanishing shocks is summarized in Proposition 5. Define $\overline{Z}_0$, $\underline{Z}_\alpha$, $\overline{Z}_\alpha$ and $\underline{Z}_1$ as satisfying, respectively,

$$\int_0^\alpha (\alpha - n)^{\frac{\rho}{\delta}} \overline{\pi}\left(\overline{Z}_0, n\right) dn = 0, \tag{17}$$

$$\int_\alpha^1 (1 - n)^{\frac{\rho}{\delta}} \underline{\pi}\left(\underline{Z}_\alpha, n\right) dn = 0, \tag{18}$$

$$\int_0^\alpha n^{\frac{\rho}{\delta}} \overline{\pi}\left(\overline{Z}_\alpha, n\right) dn = 0 \tag{19}$$

[19] The characterization of the planner's solution under asymmetric network effects follows the same steps presented in this section and is available upon request.

and

$$\int_{\alpha}^{1}(n-\alpha)^{\frac{\rho}{\delta}}\,\underline{\pi}\left(\underline{Z}_1,n\right)dn=0. \tag{20}$$

Proposition 5. *In the limiting case in which $\mu,\sigma\to 0$, thresholds are computed as follows:*

1. *(Large heterogeneity) Case $\overline{P}(n)<\underline{O}(n)\forall n$*
   (a) *Type-$\overline{q}$ agents' threshold:*
      i. *If $\overline{Z}_0<\underline{Z}_{\alpha}$, then*
      - *$\forall n_0\geq\alpha$, $\overline{Z}(n_0)=\overline{P}(n_0)$ and satisfies*

$$\int_0^{n_0} n^{\frac{\rho}{\delta}}\,\overline{\pi}\left(\overline{Z},n\right)dn=0; \tag{21}$$

      - *$\forall n_0<\alpha$, $\overline{Z}(n_0)$ satisfies*

$$\int_0^{n_0}\left(\frac{n}{n_0}\right)^{\frac{\rho}{\delta}}\overline{\pi}(\overline{Z},n)dn+\int_{n_0}^{\alpha}\left(\frac{\alpha-n}{\alpha-n_0}\right)^{\frac{\rho}{\delta}}\overline{\pi}(\overline{Z},n)dn=0. \tag{22}$$

      ii. *If $\overline{Z}_0\geq\underline{Z}_{\alpha}$, then*
      - *$\forall n_0\geq\alpha$, $\overline{Z}(n_0)=\overline{P}(n_0)$ and satisfies (21);*
      - *$\forall n_0\in(n',\alpha)$, $\overline{Z}(n_0)$ satisfies (22);*
      - *$\forall n_0\leq n'$, $\overline{Z}(n_0)$ is the solution to the system*

$$\int_0^{n_0}\left(\frac{n}{n_0}\right)^{\frac{\rho}{\delta}}\overline{\pi}\left(\overline{Z},n\right)dn+\int_{n_0}^{n_{\bar{t}}}\left(\frac{\alpha-n}{\alpha-n_0}\right)^{\frac{\rho}{\delta}}\overline{\pi}\left(\overline{Z},n\right)dn+\left(\frac{\alpha-n_{\bar{t}}}{1-n_{\bar{t}}}\right)^{\frac{\rho+\delta}{\delta}}\int_{n_{\bar{t}}}^{1}\left(\frac{1-n}{\alpha-n_0}\right)^{\frac{\rho}{\delta}}\times\overline{\pi}\left(\overline{Z},n\right)dn=0, \tag{23}$$

$$\int_{n_{\bar{t}}}^{1}(1-n)^{\frac{\rho}{\delta}}\underline{\pi}(\overline{Z},n)dn=0;$$

      - *n' is the value satisfying $\overline{Z}(n')=\underline{Z}(\alpha)$.*

   (b) *Type-$\underline{q}$ agents' threshold:*
      i. *If $\overline{Z}_{\alpha}<\underline{Z}_1$, then*
      - *$\forall n_0\leq\alpha$, $\underline{Z}(n_0)=\underline{O}(n_0)$ and satisfies*

$$\int_{n_0}^{1}(1-n)^{\frac{\rho}{\delta}}\,\underline{\pi}\left(\underline{Z},n\right)dn=0; \tag{24}$$

      - *$\forall n_0>\alpha$, $\underline{Z}(n_0)$ satisfies*

$$\int_{\alpha}^{n_0}\left(\frac{n-\alpha}{n_0-\alpha}\right)^{\frac{\rho}{\delta}}\underline{\pi}\left(\underline{Z},n\right)dn+\int_{n_0}^{1}\left(\frac{1-n}{1-n_0}\right)^{\frac{\rho}{\delta}}\underline{\pi}\left(\underline{Z},n\right)dn=0. \tag{25}$$

      ii. *If $\overline{Z}_{\alpha}\geq\underline{Z}_1$, then*
      - *$\forall n_0\leq\alpha$, $\underline{Z}(n_0)=\underline{O}(n_0)$ and satisfies (24);*
      - *$\forall n_0\in\left(\alpha,n''\right)$, $\underline{Z}(n_0)$ satisfies (25);*
      - *$\forall n_0\geq n''$, $\underline{Z}(n_0)$ is the solution to the system*

$$\int_0^{n_{\underline{t}}}\left(\frac{n}{n_0-\alpha}\right)^{\frac{\rho}{\delta}}\left(\frac{n_{\underline{t}}-\alpha}{n_{\underline{t}}}\right)^{\frac{\rho+\delta}{\delta}}\underline{\pi}\left(\underline{Z},n\right)dn+\int_{n_{\underline{t}}}^{n_0}\left(\frac{n-\alpha}{n_0-\alpha}\right)^{\frac{\rho}{\delta}}\underline{\pi}\left(\underline{Z},n\right)dn+\int_{n_0}^{1}\left(\frac{1-n}{1-n_0}\right)^{\frac{\rho}{\delta}}\underline{\pi}\left(\underline{Z},n\right)dn=0, \tag{26}$$

$$\int_0^{n_{\underline{t}}} n^{\frac{\rho}{\delta}}\overline{\pi}(\underline{Z},n)dn=0;$$

      - *n'' is the value satisfying $\underline{Z}(n'')=\overline{Z}(\alpha)$.*

2. *(Not so large heterogeneity) Case $\overline{P}(n)\geq\underline{O}(n)\forall n$*
   (a) *Type-$\overline{q}$ agents' threshold: For each n_0 such that $\overline{Z}(n_0)\neq\underline{Z}(n_0)$,*
      - *if $n_0>\alpha$, $\overline{Z}(n_0)=\overline{P}(n_0)$ and satisfies (21);*
      - *if $n_0<\alpha$, $\overline{Z}(n_0)$ is the solution to (23).*

   (b) *Type-$\underline{q}$ agents' threshold: For each n_0 such that $\underline{Z}(n_0)\neq\overline{Z}(n_0)$,*
      - *if $n_0>\alpha$, $\underline{Z}(n_0)$ is the solution to (26);*
      - *if $n_0<\alpha$, $\underline{Z}(n_0)=\underline{O}(n_0)$ and satisfies (24).*

Proof. Suppose $\mu,\sigma\to 0$.

Large heterogeneity Suppose $\overline{P}(n)<\underline{O}(n)\forall n$. By Proposition 2, we know that $\overline{Z}(n)<\underline{Z}(n)\forall n$. So, we can apply Lemma 2 to compute the bifurcation probabilities at all points along the thresholds. For all $n_0\geq\alpha$, a type-$\overline{q}$ agent's belief over n_t in equilibrium is exactly the belief we assume to compute her upper dominance region boundary, i.e., she assigns probability one to n going down at the maximum rate, $\dot{n}_t=-\delta n_t$. Thus, type-$\overline{q}$ agents' threshold $\forall n_0\geq\alpha$ is given by Eq. (2). Performing a change of variables such that $n=n_t^{\downarrow}=n_0e^{-\delta t}$, we obtain Eq. (21) in 1.(a)i. Likewise, for all $n_0\leq\alpha$, a type-$\underline{q}$ agent's belief over n_t in equilibrium is the more optimistic as possible, so her threshold is given by Eq. (3) $\forall n\leq\alpha$. A change of variables such that $n=n_t^{\uparrow}=1-(1-n_0)e^{-\delta t}$ gives us Eq. (24) in 1.(b)i.

We still have to compute type-$\overline{q}$ threshold below α and type-$\underline{q}$ threshold above α in the case of large heterogeneity ($\overline{P}$ to the left of $\underline{O}$). Consider a high-type agent. Let us assume, for now, that the equilibrium is such that the distance between the thresholds of the two types of agents is big enough so that $\overline{Z}(0)<\underline{Z}(\alpha)$ as in Fig. 13. We will show that this is the case whenever $\overline{Z}_0<\underline{Z}_{\alpha}$.

Notice that at any point on $\overline{Z}$ below α, if the system bifurcates up, n_t will grow towards α and it will never reach the low-type threshold. Consider an agent $i\in[0,\alpha]$ at some point (θ_0,n_0) with $\theta_0=\overline{Z}(n_0)$ and $n_0<\alpha$, i.e., at some point on her threshold below α. Equating her expected payoff to zero, we have

$$\underbrace{\frac{(\alpha-n_0)}{\alpha}}_{P(\text{up})}\int_0^{\infty}e^{-(\rho+\delta)t}\overline{\pi}(\overline{Z},\underbrace{\alpha-(\alpha-n_0)e^{-\delta t}}_{n_t\text{ growing towards }\alpha})dt+\underbrace{\frac{n_0}{\alpha}}_{P(\text{down})}\int_0^{\infty}e^{-(\rho+\delta)t}\overline{\pi}(\overline{Z},\underbrace{n_0e^{-\delta t}}_{n_t\text{ falling}})dt=0. \tag{27}$$

The first term of the sum is the probability of an upward bifurcation times the discounted payoff when the agent expects n_t to grow until it approaches α. The second one is the probability of a downward bifurcation times the discounted payoff when the agent

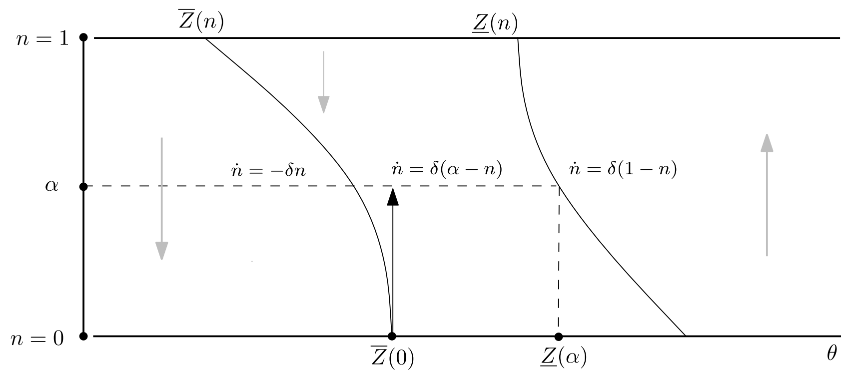

Fig. 13. Very large heterogeneity.

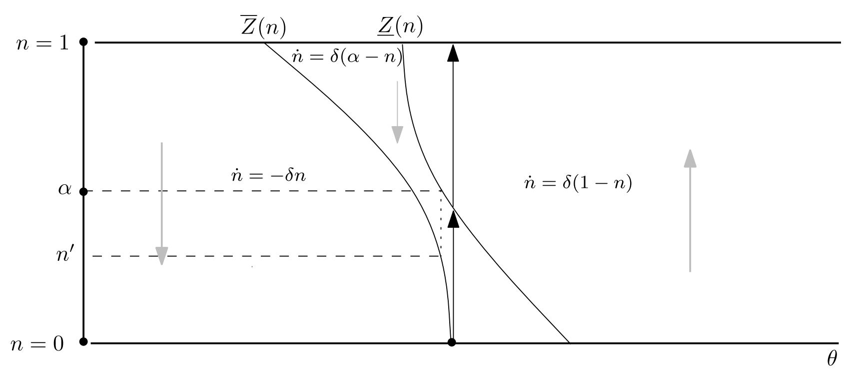

Fig. 14. Large heterogeneity.

expects n_t to decrease towards zero. Integrating by substitution the two terms in the equation above (letting $n=\alpha-(\alpha-n_0)e^{-\delta t}$ in the first and $n=n_0e^{-\delta t}$ in the second integral), we have Eq. (22) in 1.(a)i. Remember we have computed this threshold assuming $\overline{Z}(0)<\underline{Z}(\alpha)$, which is only the case when the value of $\overline{Z}(0)$ obtained using the expression above is smaller than $\underline{Z}(\alpha)$. Evaluating the equation above at $n_0=0$ and operating a change of variables such that $n=\alpha-\alpha e^{-\delta t}$, we have the expression for $\overline{Z}_0$ in Eq. (17). Thus, whenever $\overline{Z}_0<\underline{Z}_\alpha$, where $\underline{Z}_\alpha\equiv\underline{Z}(\alpha)$, the high-type threshold is in fact given by Eq. (22) for all $n<\alpha$.

Now, assume instead that $\overline{Z}_0\geq\underline{Z}_\alpha$. In that case, a high-type agent making a choice at some point on her threshold needs to take into account that, depending on the initial state (θ_0,n_0), the system may bifurcate up but not only towards $n_t=\alpha$. For some (low) values of n_0, following an upward bifurcation, n_t will grow at a lower rate (all high-type agents play 1 but all low types play 0) until the system crosses the low-type threshold, and thereafter everyone who gets the chance to revise theirs actions will choose 1. Fig. 14 illustrates this case.

Define n' as satisfying $\overline{Z}(n')=\underline{Z}(\alpha)$. For all $n_0\in(n',\alpha)$, $\overline{Z}(n_0)$ is still given by Eq. (22). But consider now an initial point (θ_0,n_0) with $\theta_0=\overline{Z}(n_0)$ and $n_0\leq n'$. In this case, the high-type threshold can be computed as

$$\underbrace{\frac{(\alpha-n_0)}{\alpha}}_{P(\text{up})}\left\{\int_0^{\bar{t}}e^{-(\rho+\delta)t}\overline{\pi}(\overline{Z},\underbrace{\alpha-(\alpha-n_0)e^{-\delta t}}_{n_t\text{ growing towards }\alpha})dt\right.$$
$$\left.+\int_{\bar{t}}^{\infty}e^{-(\rho+\delta)t}\overline{\pi}(\overline{Z}.\underbrace{1-(1-n_{\bar{t}})e^{-\delta(t-\bar{t})}}_{n_t\text{ growing towards }1})dt\right\}$$

$$\underbrace{\frac{n_0}{\alpha}}_{P(\text{down})}\int_0^{\infty}e^{-(\rho+\delta)t}\overline{\pi}(\overline{Z},\underbrace{n_0e^{-\delta t}}_{n_t\text{ falling}})dt=0,\tag{28}$$

where $\bar{t}$ denote the time at which the economy reaches $\underline{Z}$ in case an upward bifurcation occurs and $n_{\bar{t}}=\underline{Z}^{-1}(\overline{Z}(n_0))$. Since $n_{\bar{t}}=\alpha-(\alpha-n_0)e^{-\delta\bar{t}}$, we have that $\bar{t}=-\frac{1}{\delta}\ln\frac{\alpha-n_{\bar{t}}}{\alpha-n_0}$. Performing a change of variables in each one of the integrals in the equation above, we get to the first line of the system in (23). The second line is equivalent to $n_{\bar{t}}=\underline{Z}^{-1}(\overline{Z}(n_0))$, using the fact that, for all $n\leq\alpha$, $\underline{Z}$ is given by Eq. (24). The low-type threshold above α when either $\overline{\overline{Z}}_\alpha<\underline{Z}_1$ or $\overline{Z}_\alpha\geq\underline{Z}_1$ is computed following analogous steps.

Small heterogeneity Now, suppose $\overline{P}(n)\geq\underline{Q}(n)\forall n$. By Proposition 2, we know that there is a neighborhood of α such that agents play according to the same strategy $Z(n)$ whenever n is in this neighborhood, but thresholds never coincide for all $n\in[0,1]$. Parts of thresholds that coincide cannot be computed using the bifurcation probabilities, since Lemma 2 cannot be used to pin down agents' beliefs. Computing the interval(s) of n such that thresholds coincide requires knowing the specific functional form of payoffs (see linear example.) For every value of n such that $\overline{Z}(n)\neq\underline{Z}(n)$, it is possible to compute the thresholds using the bifurcation probabilities in Lemma 2, as in the case of large heterogeneity. If $n_0\geq\alpha$, $\overline{Z}(n_0)=\overline{P}(n_0)$ and thus satisfies (21); $\underline{Z}(n_0)$ is the solution to (26), given that whenever the system bifurcates down, it eventually crosses $\overline{Z}$ and decreases towards $n=0$. If $n_0\leq\alpha$, $\overline{Z}(n_0)$ is the solution to (23) and $\underline{Z}(n_0)=\underline{Q}(n_0)$ and thus satisfies (24). □

A.2. Vanishing frictions

The next proposition characterize the equilibrium in the limit as timing frictions shrink.

Proposition 6. *In the limit as frictions vanish ($\delta \to \infty$), the equilibrium is characterized by thresholds $\overline{Z}^*(n_0)$ and $\underline{Z}^*(n_0)$ computed as follows. The dominance regions' boundaries of interest now satisfy*

$$\int_0^{n_0} \overline{\pi}(\overline{P}^*, n)dn = 0$$

and

$$\int_{n_0}^{1} \underline{\pi}(\underline{O}^*, n)dn = 0.$$

1. *(Large heterogeneity) Case $\overline{P}^*(n) < \underline{O}^*(n) \forall n$*
   (a) *Type-$\overline{q}$ agents' threshold:*
   - *$\forall n_0 \geq \alpha$, $\overline{Z}^*(n_0)$ satisfies*
   $$\int_0^{n_0} \overline{\pi}\left(\overline{Z}^*, n\right) dn = 0; \tag{29}$$
   - *$\forall n < \alpha$, $\overline{Z}^*(n_0)$ satisfies*
   $$\int_0^{\alpha} \overline{\pi}(\overline{Z}^*, n)dn = 0, \tag{30}$$
   which is independent of n_0.

   (b) *Type-$\underline{q}$ agents' threshold:*
   - *$\forall n_0 \leq \alpha$, $\underline{Z}^*(n_0)$ satisfies*
   $$\int_{n_0}^{1} \underline{\pi}\left(\underline{Z}^*, n\right) dn = 0; \tag{31}$$
   - *$\forall n_0 > \alpha$, $\underline{Z}^*(n_0)$ satisfies*
   $$\int_{\alpha}^{1} \underline{\pi}\left(\underline{Z}^*, n\right) dn = 0, \tag{32}$$
   which is independent of n_0.

2. *(Not so large heterogeneity) Case $\overline{P}^*(n) \geq \underline{O}^*(n) \forall n$*
   (a) *Type-$\overline{q}$ agents' threshold: For each n_0 such that $\overline{Z}^*(n_0) \neq \underline{Z}^*(n_0)$,*
   - *if $n_0 > \alpha$, $\overline{Z}^*(n_0) = \overline{P}^*(n_0)$ and satisfies (29);*
   - *if $n_0 < \alpha$, $\overline{Z}^*(n_0)$ is the solution to the system*
   $$\int_0^{n_{\overline{t}}} \overline{\pi}\left(\overline{Z}^*, n\right) dn + \left(\frac{\alpha - n_{\overline{t}}}{1 - n_{\overline{t}}}\right)^{\frac{\rho+\delta}{\delta}} \int_{n_{\overline{t}}}^{1} \overline{\pi}\left(\overline{Z}^*, n\right) dn = 0, \tag{33}$$
   $$\int_{n_{\overline{t}}}^{1} \underline{\pi}(\overline{Z}^*, n)dn = 0.$$
   Notice that $\overline{Z}^*$ is a vertical line for $n_0 < \alpha$.

   (b) *Type-$\underline{q}$ agents' threshold: For each n_0 such that $\underline{Z}^*(n_0) \neq \overline{Z}^*(n_0)$,*
   - *if $n_0 < \alpha$, $\underline{Z}^*(n_0) = \underline{O}^*(n_0)$ and satisfies (31);*
   - *if $n_0 > \alpha$, $\underline{Z}^*(n_0)$ is the solution to the system*
   $$\left(\frac{n_{\underline{t}} - \alpha}{n_{\underline{t}}}\right)^{\frac{\rho+\delta}{\delta}} \int_0^{n_{\underline{t}}} \underline{\pi}\left(\underline{Z}^*, n\right) dn + \int_{n_{\underline{t}}}^{1} \underline{\pi}\left(\underline{Z}^*, n\right) dn = 0, \tag{34}$$
   $$\int_0^{n_{\underline{t}}} \overline{\pi}(\underline{Z}^*, n)dn = 0.$$
   Notice that $\underline{Z}^*$ is a vertical line for $n_0 > \alpha$.

If $\overline{\pi}(\theta, n) = \pi(\theta, n) + \overline{\varepsilon}$ and $\underline{\pi}(\theta, n) = \pi(\theta, n) + \underline{\varepsilon}$, with $\overline{\varepsilon} > \underline{\varepsilon}$, the equilibrium is fully characterized as follows: Define $\hat{\varepsilon} \equiv \alpha\overline{\varepsilon} + (1-\alpha)\underline{\varepsilon}$ and $\hat{z}^*$ as satisfying $\int_0^1 \pi\left(\hat{z}^*, n\right) dn = -\hat{\varepsilon}$. $\forall n_0 \geq n_{\overline{t}}$, $\underline{Z}^*(n_0) = \hat{z}^*$ and $\forall n_0 < n_{\overline{t}}$, $\underline{Z}^*(n_0)$ is given by Eq. (31). $\forall n_0 \leq n_{\underline{t}}$, $\overline{Z}^*(n_0) = \hat{z}^*$ and $\forall n_0 > n_{\underline{t}}$, $\overline{Z}^*(n_0)$ is given by Eq. (29). $n_{\underline{t}}$ and $n_{\overline{t}}$ satisfy $\int_0^{n_{\underline{t}}} \pi(\hat{z}^*, n)dn = -n_{\underline{t}}\overline{\varepsilon}$ and $\int_{n_{\overline{t}}}^1 \pi(\hat{z}^*, n)dn = -(1-n_{\overline{t}})\underline{\varepsilon}$, respectively.

Proof. We know that, for any t, $(\theta_t - \theta_0) \sim \mathcal{N}\left(\mu t, \sigma^2 t\right)$. If we rescale time as $\tilde{t} = t/\delta$, as in Theorem 3 in Frankel and Pauzner (2000), we can apply Proposition 5 in order to compute the equilibrium, given that taking the limit as $\delta \to \infty$ is analogous to assuming $\mu, \sigma \to 0$. Proposition 5 and the fact that $\lim_{\delta\to\infty} \frac{\rho}{\delta} = 0$ and $\lim_{\delta\to\infty} \frac{\rho+\delta}{\delta} = 1$ give us the desired result. Notice that it is not necessary to divide the analysis of the large heterogeneity case in two: since the lower part of the high-type threshold and the upper part of the low-type threshold are vertical, whenever $\underline{O}$ is to the left of $\overline{P}$, conditions $\overline{Z}_0 < \underline{Z}_\alpha$ and $\overline{Z}_\alpha < \underline{Z}_1$ (with $\rho/\delta \to 0$) are automatically satisfied.

Finally, suppose $\underline{\pi}(\theta, n) = \pi(\theta, n) + \underline{\varepsilon}$, $\overline{\pi}(\theta, n) = \pi(\theta, n) + \overline{\varepsilon}$ and not too large heterogeneity. Let $\hat{\overline{Z}}$ be the solution to system (33) (which gives us the high-type threshold for low values of n) and $\hat{\underline{Z}}$ be the solution to system (34) (which gives us the low-type threshold for high values of n). Solving the two systems, we find $\int_0^1 \pi\left(\hat{\overline{Z}}, n\right) dn = -\left[\alpha\overline{\varepsilon} + (1-\alpha)\underline{\varepsilon}\right] \equiv -\hat{\varepsilon}$ and $\int_0^1 \pi\left(\hat{\underline{Z}}, n\right) dn = -\hat{\varepsilon}$, which implies $\hat{\overline{Z}} = \hat{\underline{Z}} = \hat{z}^*$. Also, we know that $\overline{Z}(n) = \underline{Z}(n)$ for n is some neighborhood of α. Since thresholds cannot be upward sloping, whenever agents play the same strategy, their threshold is also given by $\hat{z}^*$. Hence, we can fully characterize the equilibrium in this particular case, which is depicted in Fig. 6. □

A.3. Low-type equilibrium threshold with linear payoffs

- If $\overline{\varepsilon} - \underline{\varepsilon} > \frac{\gamma[\delta+\rho(1-\alpha)]}{\rho+2\delta}$,
$$\underline{Z} = \begin{cases} -\underline{\varepsilon} - \dfrac{\gamma\delta(1+\alpha)}{\rho+2\delta} - \dfrac{\gamma\rho}{\rho+2\delta}n & \text{if } n > \alpha, \\ -\underline{\varepsilon} - \dfrac{\gamma\delta}{\rho+2\delta} - \dfrac{\gamma(\rho+\delta)}{\rho+2\delta}n & \text{if } n \leq \alpha. \end{cases} \tag{35}$$
- If $\frac{\gamma[\delta+\rho(1-\alpha)]}{\rho+2\delta} \leq \overline{\varepsilon} - \underline{\varepsilon} < \frac{\gamma\delta}{\rho+2\delta}$, $\underline{Z}$ is given by (35) $\forall n \leq n''$ and otherwise it satisfies
$$\frac{(1-n)}{1-\alpha}\int_0^\infty e^{-(\rho+\delta)t}\left[\underline{Z} + \underline{\varepsilon} + \gamma\left(1-(1-n)e^{-\delta t}\right)\right]dt + \frac{(n-\alpha)}{1-\alpha}\left\{\int_0^{\underline{t}} e^{-(\rho+\delta)t}\left[\underline{Z} + \underline{\varepsilon} + \gamma\left(\alpha + (n-\alpha)e^{-\delta t}\right)\right] + \int_{\underline{t}}^\infty e^{-(\rho+\delta)t}\left[\underline{Z} + \underline{\varepsilon} + \gamma\left(n_{\underline{t}}e^{-\delta(t-\underline{t})}\right)\right]dt\right\} = 0, \tag{36}$$
where $\underline{t} = -\frac{1}{\delta}\ln\frac{n_{\underline{t}}-\alpha}{n_0-\alpha}$ and $n_{\underline{t}} = -\left(\underline{Z} + \overline{\varepsilon}\right)(\rho+2\delta)/\gamma(\rho+\delta)$. Integrating by substitution, we can express $\underline{Z}(n) \forall n \leq n''$ as satisfying
$$(\rho+2\delta)(\underline{Z}+\underline{\varepsilon}) + \gamma\rho n_0 + \gamma\delta(1+\alpha) - \frac{\alpha}{(1-\alpha)}\gamma\delta\left(\frac{1}{n_0-\alpha}\right)^{\frac{\rho}{\delta}}\left[-\frac{(\underline{Z}+\overline{\varepsilon})(\rho+2\delta)}{\gamma(\rho+\delta)} - \alpha\right]^{\frac{\rho+\delta}{\delta}} = 0.$$
n'' is the value satisfying $\underline{Z}(n'') = \overline{Z}(\alpha)$, which results in $n'' = \alpha + \frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)-\gamma\delta}{\gamma\rho}$.

- If $\overline{\varepsilon}-\underline{\varepsilon}\le\frac{\gamma\delta}{\rho+2\delta}$, we get the following characterization. For all $n\le\hat{n}$,

$$\underline{Z}=-\underline{\varepsilon}-\frac{\gamma\delta}{\rho+2\delta}-\frac{\gamma(\rho+\delta)}{\rho+2\delta}n,$$

and for all $n\ge\hat{\hat{n}}$, $\underline{Z}$ satisfies (36). By intersecting the different types' distinct thresholds, we get that

$$\hat{n}=\alpha\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)}{\gamma\delta}<\alpha$$

and

$$\hat{\hat{n}}=1-(1-\alpha)\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)}{\gamma\delta}>\alpha.$$

For all $n\in\left[\hat{n},\hat{\hat{n}}\right]$, the two types of agents play according to the same threshold, which cannot be computed using the bifurcation probabilities. However, it is possible to compute the coinciding part of the thresholds building on the planner's solution presented in Proposition 7. The planner's solution is equivalent to the solution of the game played by agents, only with different payoffs. Appendix A.4 shows that the coinciding part of the planner's solution based on the decision rule

$$\mathbb{E}\int_0^\infty e^{-(\rho+\delta)t}\left(\theta_t-\frac{\gamma}{2}+2\gamma n_t+\varepsilon_i\right)dt>0$$

is Z^P as given in (37). Agents, however, choose according to

$$\mathbb{E}\int_0^\infty e^{-(\rho+\delta)t}\left(\theta_t+\gamma n_t+\varepsilon_i\right)dt>0.$$

We can, thus, recover the coinciding part of agents' thresholds. Agents' solution in a world with fundamentals $\tilde{\theta}=\theta-\gamma/2$ and externality parameter $\tilde{\gamma}=2\gamma$ must be the same as the solution to a planner's problem with fundamentals θ and externality parameter γ. Performing this transformation, we get that, for all $n\in\left[\hat{n},\hat{\hat{n}}\right]$,

$$\underline{Z}=\overline{Z}=Z\equiv-\hat{\varepsilon}-\frac{\gamma\delta}{\rho+2\delta}-\frac{\gamma\rho n}{\rho+2\delta}.$$

It is interesting to note that this is exactly the threshold we would have if agents were all of a single intermediate type, with preference parameter $\hat{\varepsilon}=\alpha\overline{\varepsilon}+(1-\alpha)\underline{\varepsilon}$.

A.4. Planner's solution with two types of agents

Proposition 7. *Consider the model with two types of agents and linear payoff functions with $\nu^0=\nu^1\equiv\nu$. Define*

$$Z^P=-\hat{\varepsilon}-\frac{\gamma\delta}{\rho+2\delta}+\frac{\gamma\rho}{2(\rho+2\delta)}-\frac{2\gamma\rho}{\rho+2\delta}n. \quad (37)$$

The planner's solution is characterized by thresholds $\overline{Z}^P$ and $\underline{Z}^P$ as follows:

1. *Planner's type-$\overline{\varepsilon}$ threshold:*

   (a) *If $\overline{\varepsilon}-\underline{\varepsilon}>\frac{2\gamma(\delta+\rho\alpha)}{\rho+2\delta}$,*

$$\overline{Z}^P=\begin{cases}-\overline{\varepsilon}+\frac{\gamma}{2}-\frac{2\gamma(\rho+\delta)}{\rho+2\delta}n & \text{if } n\ge\alpha,\\ -\overline{\varepsilon}+\frac{\gamma}{2}-\frac{2\alpha\gamma\delta}{\rho+2\delta}-\frac{2\gamma\rho}{\rho+2\delta}n & \text{if } n<\alpha.\end{cases} \quad (38)$$

   (b) *If $\frac{2\gamma\delta}{\rho+2\delta}<\overline{\varepsilon}-\underline{\varepsilon}\le\frac{2\gamma(\delta+\alpha\rho)}{\rho+2\delta}$,*

   - *$\forall n\ge n'^P\equiv\alpha-\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)-2\gamma\delta}{2\gamma\rho}$, $\overline{Z}^P$ is given by Eq. (38);*
   - *$\forall n<n'^P$, it satisfies*

$$(\rho+2\delta)\left(\overline{Z}^P+\overline{\varepsilon}-\gamma/2\right)+2\gamma\delta\alpha+2\gamma\rho n_0+2\gamma\delta\frac{(1-\alpha)}{\alpha}\left(\frac{1}{\alpha-n_0}\right)^{\frac{\rho}{\delta}}\times\left(\alpha+\frac{(\overline{Z}^P+\underline{\varepsilon}-\gamma/2)(\rho+2\delta)+2\gamma\delta}{2\gamma(\rho+\delta)}\right)^{\frac{\rho+\delta}{\delta}}=0; \quad (39)$$

   (c) *If $\overline{\varepsilon}-\underline{\varepsilon}\le\frac{2\gamma\delta}{\rho+2\delta}$,*

   - *$\forall n\le\hat{n}^P\equiv\alpha\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)}{2\gamma\delta}$, $\overline{Z}^P$ satisfies (39);*
   - *$\forall n\ge\hat{\hat{n}}^P\equiv1-(1-\alpha)\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)}{2\gamma\delta}$,*

$$\overline{Z}^P=-\overline{\varepsilon}+\frac{\gamma}{2}-\frac{2\gamma(\rho+\delta)}{\rho+2\delta}n.$$

   - *$\forall n\in[\hat{n}^P,\hat{\hat{n}}^P]$, $\overline{Z}^P=Z^P$.*

2. *Planner's type-$\underline{\varepsilon}$ threshold:*

   (a) *If $\overline{\varepsilon}-\underline{\varepsilon}>\frac{2\gamma(\delta+\rho(1-\alpha))}{\rho+2\delta}$,*

$$\underline{Z}^P=\begin{cases}-\underline{\varepsilon}+\frac{\gamma}{2}-\frac{2\gamma\delta(1+\alpha)}{\rho+2\delta}-\frac{2\gamma\rho}{\rho+2\delta}n & \text{if } n>\alpha,\\ -\underline{\varepsilon}+\frac{\gamma}{2}-\frac{2\gamma\delta}{\rho+2\delta}-\frac{2\gamma(\rho+\delta)}{\rho+2\delta}n & \text{if } n\le\alpha.\end{cases} \quad (40)$$

   (b) *If $\frac{2\gamma\delta}{\rho+2\delta}<\overline{\varepsilon}-\underline{\varepsilon}\le\frac{2\gamma(\delta+\rho(1-\alpha))}{\rho+2\delta}$,*

   - *$\forall n\le n''^P=\alpha+\frac{(\overline{\varepsilon}-\underline{\varepsilon})(\rho+2\delta)-2\gamma\delta}{2\gamma\rho}$, $\underline{Z}^P$ is given by Eq. (40);*
   - *$\forall n>n''^P$, $\underline{Z}^P$ satisfies*

$$(\rho+2\delta)(\underline{Z}^P+\underline{\varepsilon}-\gamma/2)+2\gamma\rho n_0+2\gamma\delta(1+\alpha)+2\gamma\delta\frac{\alpha}{(1-\alpha)}\left(\frac{1}{n_0-\alpha}\right)^{\frac{\rho}{\delta}}\times\left(\alpha+\frac{(\underline{Z}^P+\overline{\varepsilon}-\gamma/2)(\rho+2\delta)}{2\gamma(\rho+\delta)}\right)^{\frac{\rho+\delta}{\delta}}=0. \quad (41)$$

   (c) *If $\overline{\varepsilon}-\underline{\varepsilon}\le\frac{2\gamma\delta}{\rho+2\delta}$,*

   - *$\forall n\ge\hat{\hat{n}}^P$, $\underline{Z}^P$ satisfies (41);*
   - *$\forall n\le\hat{n}^P$,*

$$\underline{Z}^P=-\overline{\varepsilon}+\frac{\gamma}{2}-\frac{2\gamma\delta}{\rho+2\delta}-\frac{2\gamma(\rho+\delta)}{\rho+2\delta}n.$$

   - *$\forall n\in[\hat{n}^P,\hat{\hat{n}}^P]$, $\underline{Z}^P=Z^P$.*

Proof. The proof of Proposition 7 follows from the discussion in Section 5. Since solving the planner's problem is equivalent to solving a game with different payoffs, the proof is equivalent to the proof of Proposition 5 if we substitute agents' flow-payoffs, $\underline{\pi}(.)$ and $\overline{\pi}(.)$, by the expressions arising from the optimality conditions: $\underline{Z}_t^P+\underline{\varepsilon}-\frac{\gamma}{2}+2\gamma n_t$ and $\overline{Z}_t^P+\overline{\varepsilon}-\frac{\gamma}{2}+2\gamma n_t$, respectively. The characterization of the coinciding thresholds in the case of small heterogeneity and $n \in [\hat{n}^P, \hat{\hat{n}}^P]$ is also possible, even though we cannot apply the bifurcation probabilities to compute the equilibrium beliefs. Notice that, when $n \in [\hat{n}^P, \hat{\hat{n}}^P]$, if there is a threshold to the right of which the planner recommends action 1 for all agents and to the left of which it recommends action 0, then an alternative way to characterize this piece of the threshold is to solve the indifference condition between recommending each action forever, that is,[20]

$$\alpha \int_0^\infty e^{-\rho t}\left[\overline{n}_t^{\uparrow}\left(\theta_1+\nu n_t^{\uparrow}+\overline{\varepsilon}\right)+(1-\overline{n}_t^{\uparrow})\left(\theta_0+\nu(1-n_t^{\uparrow})\right)\right]dt$$
$$+(1-\alpha)\int_0^\infty e^{-\rho t}\left[\underline{n}_t^{\uparrow}\left(\theta_1+\nu n_t^{\uparrow}+\underline{\varepsilon}\right)\right.$$
$$\left.+\ (1-\underline{n}_t^{\uparrow})\left(\theta_0+\nu(1-n_t^{\uparrow})\right)\right]dt$$
$$=\alpha\int_0^\infty e^{-\rho t}\left[\overline{n}_t^{\downarrow}\left(\theta_1+\nu n_t^{\downarrow}+\overline{\varepsilon}\right)\right.$$
$$\left.+\ (1-\overline{n}_t^{\downarrow})\left(\theta_0+\nu(1-n_t^{\downarrow})\right)\right]dt$$
$$+(1-\alpha)\int_0^\infty e^{-\rho t}\left[\overline{n}_t^{\downarrow}\left(\theta_1+\nu n_t^{\downarrow}+\underline{\varepsilon}\right)\right.$$
$$\left.+\ (1-\overline{n}_t^{\downarrow})\left(\theta_0+\nu(1-n_t^{\downarrow})\right)\right]dt.$$

Using the definition $\theta = \theta^1 - \theta^0 - \nu$, this indifference condition simplifies to

$$\int_0^\infty e^{-\rho t}\left[n_t^{\uparrow}\theta+\gamma\left(n_t^{\uparrow}\right)^2-\frac{\gamma}{2}n_t^{\uparrow}+(1-e^{-\delta t})\hat{\varepsilon}\right]dt$$
$$=\int_0^\infty e^{-\rho t}\left[n_t^{\downarrow}\theta+\gamma\left(n_t^{\downarrow}\right)^2-\nu n_t^{\downarrow}\right]dt. \quad (42)$$

The coinciding part of the planner's threshold is, thus, the value of θ satisfying (42), which is Z^P as given in Eq. (37). □

Appendix B. Proofs

B.1. Proof of Proposition 1

The proof of equilibrium uniqueness follows an analogous reasoning as in the case of identical individuals (Frankel and Pauzner, 2000). Consider a type-q agent at some point on P_q. She is indifferent between actions 0 and 1 under the belief that everyone called upon choosing an action while she is committed to her choice will pick 0 under any circumstances. But when θ moves stochastically, there is always the possibility that it will spend some time to the right of some players' dominance region boundaries. Notice that even if q is such that P_q is the leftmost upper boundary (say P_3 in Fig. 1), she cannot expect every other player to choose 0 under any circumstances while she is committed to her choice. If θ moves slightly to the right, it will be strictly dominant for type-q agents to pick 1, and thus a fraction α_q of the agents that get the chance will not choose 0. The most pessimistic (regarding the path of n) belief that agents can hold consistent with the dominance regions is that each type-q agent plays 1 when to the right of P_q, and 0 when to the left of it. In other words, agents do not play strictly dominated strategies. Under this (more optimistic) new belief, the agent on P_q is not indifferent anymore, but strictly preferring to play 1. To make her indifferent, we must lower θ. We can then construct for each type q a new boundary P_q^2 (to the left of P_q), to the right of which a type-q player chooses 1 when she expects all other agents to play according to $(P_q)_{q\in\mathcal{Q}}$. This procedure can be repeated *ad infinitum*. At each round, we look for the curve P_q^k on which a type-q player has zero discounted payoff when assuming that other agents play according to $(P_q^{k-1})_{q\in\mathcal{Q}}$. Denote the limit of this sequence by $(P_q^\infty)_{q\in\mathcal{Q}}$. Notice that each agent i playing according to $P_{q(i)}^\infty$ is, in fact, an equilibrium: if she expects others to play according to $(P_q^\infty)_{q\in\mathcal{Q}}$, her best response is to play according to $P_{q(i)}^\infty$ (see Fig. 15).

We now turn to a different iterative process starting from the lower dominance regions. Let $\left(P_q^{\lambda_0}\right)_{q\in\mathcal{Q}}$ be translations of the curves P_q^∞ to the left by an amount λ_0. Fix λ_0 as the smallest distance such that all translations lie completely on the lower dominance region of each corresponding type. Fig. 16 exemplifies this step.

Now, construct for each type a new curve $P_q^{\lambda_1}$ as the rightmost translation of $P_q^{\lambda_0}$ to the left of which each type-q agent must play 0 if they expect others to play according to $\left(P_q^{\lambda_0}\right)_{q\in\mathcal{Q}}$.[21] Let $P_q^{\lambda_\infty}$ be the limit of this sequence, for each q. There is at least one point in some $P_q^{\lambda_\infty}$ curve on which a type-q agent is indifferent between the two networks, otherwise iterations would not have stopped. Without loss of generality, suppose there is a point of indifference in $P_1^{\lambda_\infty}$ and name it p. Let p' denote the point on P_1^∞ at the same height as p. If we establish that p and p' coincide, we show that the whole curves coincide and, since we have translated all curves by the same λ's, $P_q^{\lambda_\infty} = P_q^\infty \forall q$, that is, the equilibrium is unique (see Fig. 17).

Let us compare two type-1 players, one receiving an opportunity to choose an action on p (expecting others to play according to the limit translations), and the other on p' (expecting others to play according to $(P_q^\infty)_{q\in\mathcal{Q}}$). Let us name those players p and p', respectively. We know that both players expect changes in the fundamentals relative to its starting point to have the same distribution. Also, since the original curves and their translations have the same shape and the pairwise distances between $P_q^{\lambda_\infty}$'s are the same as the distances between P_q^∞'s (each round, we have translated all curves by the same λ), for a given path of the fundamental, they both expect the same dynamics for n_t (computed as in Lemma 1).[22] If $\lambda_\infty > 0$, we get a contradiction: the two players expect the same relative dynamics for the (θ_t, n_t) system and the θ that p' expects at all times exceeds the θ that the agent on p expects, thus they cannot both have zero payoff. Then, $\lambda_\infty = 0$, that is, the points p and p' must coincide. The equilibrium is unique and it is characterized by thresholds $\left(Z_q^*\right)_{q\in\mathcal{Q}}$, where $Z_q^* \equiv P_q^\infty$.

We can show these thresholds are downward sloping by induction. Under the most pessimistic beliefs possible, a type-q agent's incentives to choose action 1 is increasing in the initial values of both n and θ, meaning that P_q is downward sloping (for all q). Under the assumption that all other agents are choosing according to downward sloping thresholds $\left(P_q^{k-1}\right)_{q\in\mathcal{Q}}$, the relative payoff of a given agent must be again increasing in both initial values of θ and n, since increases in either of these values would make the system spend more time to the right of other agents' thresholds for any given path of the Brownian motion, and thus ${P_q}^k$ must also be downward sloping. □

Lemma 3 (*Used in the Proof of Proposition 1*). *Suppose agents play according to thresholds* $\left\{P_q^\infty\right\}_{q\in\mathcal{Q}}$ *or* $\left\{P_q^{\lambda_\infty}\right\}_{q\in\mathcal{Q}}$, *as defined in the proof*

[20] If for a given n the planner chooses action 1 both types, after n starts increasing action 1 must be strictly preferred by the planner. The same reasoning holds for action 0.

[21] Note that what we are doing is eliminating strictly dominated strategies once again, but we are not necessarily eliminating *all* dominated strategies each round.

[22] Lemma 3 proves that the dynamical system for $\partial n_t/\partial t$ presented in Lemma 1 has an unique solution.

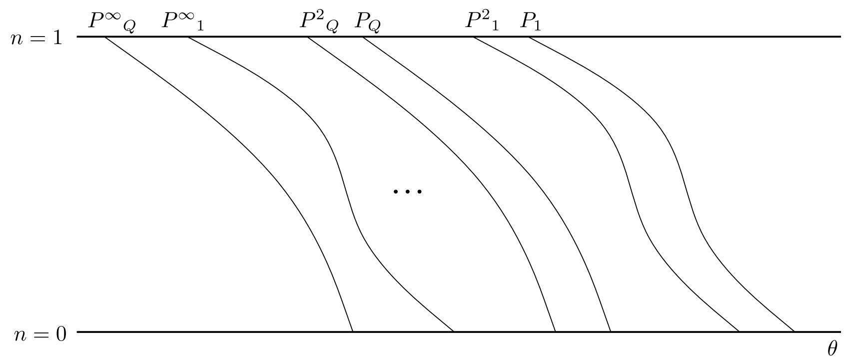

Fig. 15. Iterative deletion of strictly dominated strategies from the upper dominance region.

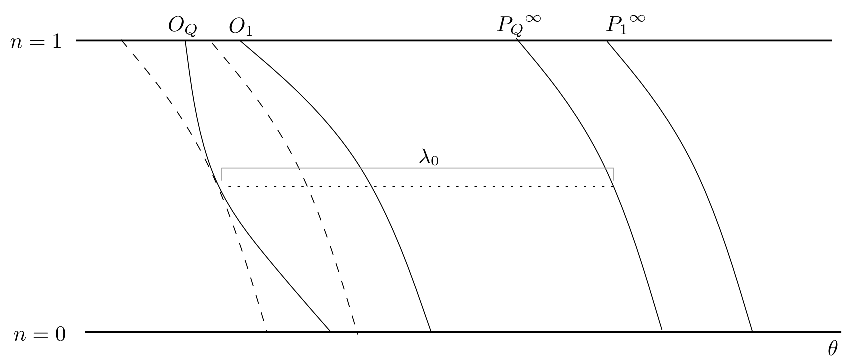

Fig. 16. Translations of P_q^{∞}.

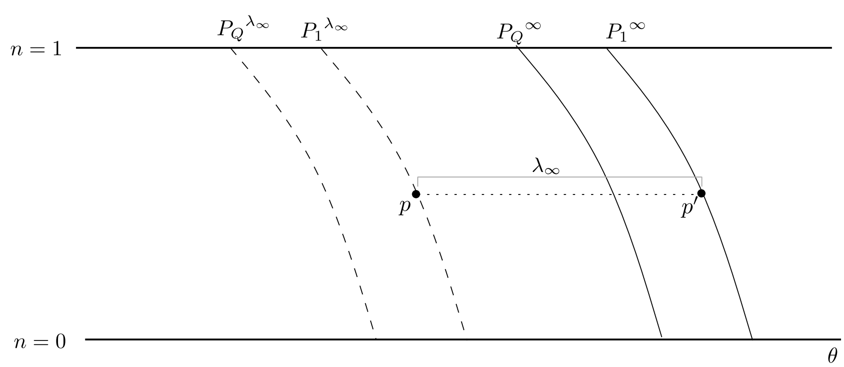

Fig. 17. Equilibrium uniqueness.

of Proposition 1. For almost every path of θ, there is an unique path for n.

Proof. The proof follows from Theorem 1 in Burdzy et al. (1998). In order to apply their result here, we must prove that all P_q^{∞}'s are Lipschitz continuous (and so are $P_q^{\lambda\infty}$'s), that is, there exists a constant c such that $|P_q^{\infty}(n) - P_q^{\infty}(n')| \leq c|n - n'|$ for all n, n' and for all $q \in \mathcal{Q}$. Notice every curve P_q^k is contained in a compact set $[0, 1] \times \left[O_q(0), P_q(1)\right]$. Thus, for each type q there are constants b_q and d_q such that, at all points (θ, n) on each P_q^k, $\partial\pi_q/\partial n < b_q$ and $\partial\pi_q/\partial\theta > d_q$, given our assumption that all functions $\pi_q(\theta, n)$ are continuously differentiable in both arguments.

First, we show that the upper dominance region boundaries $\left\{P_q\right\}_{q\in\mathcal{Q}}$ are Lipschitz. Fix an arbitrary q and consider two points along P_q, $(\tilde{\theta}, \tilde{n})$ and (θ', n'), with $\tilde{\theta} > \theta'$ and $\tilde{n} < n'$. Let us compare, for any path $(\theta_t)_{t\geq 0}$ of the Brownian motion starting at $\theta_0 = \tilde{\theta}$, the payoff of a type-$q$ agent at $(\tilde{\theta}, \tilde{n})$ to her payoff at (θ', n') when the Brownian motion is $(\theta_t + \theta' - \tilde{\theta})_{t\geq 0}$ (that is, under the same realization of shocks to θ). At all future dates, the difference in fundamentals is constant at $\tilde{\theta} - \theta'$. Since we compute P_q assuming the worst belief possible, the difference in future values of n is $(n' - n)e^{-\delta(\tau - t)} < (n' - n)$. This implies that the difference in (relative) flow payoffs at any future date when the system starts at $(\tilde{\theta}, \tilde{n})$ in comparison to when it starts at (θ', n') is always greater than $d_q(\tilde{\theta} - \theta') - b_q(n' - \tilde{n})$. For type-$q$ agents to be indifferent at both points, it must be that $d_q(\tilde{\theta} - \theta') - b_q(n' - \tilde{n}) \leq 0$,[23] or $(\tilde{\theta} - \theta')/(n' - \tilde{n}) \leq b_q/d_q = c_q$, which means P_q is Lipschitz with constant c_q. Thus, defining $c \equiv \max\left\{c_q\right\}_{q\in\mathcal{Q}}$, we have that all P_q's are Lipschitz with constant c.

We can now show by induction that all $\left\{P_q^k\right\}_{q\in\mathcal{Q}}$ curves are Lipschitz with constant c. Let $\left\{P_q^{k-1}\right\}_{q\in\mathcal{Q}}$ be Lipschitz with constant

23 If differences in flow payoffs were positive at all future dates, they would not integrate to zero.

c and suppose by contradiction that $P^k_{q'}$ is not (for an arbitrary q'). Then, there must be points $A = (\theta', n')$ and $B = (\tilde{\theta}, \tilde{n})$ along $P^k_{q'}$ with $\tilde{\theta} > \theta'$ and $\tilde{n} < n'$ such that $(\tilde{\theta} - \theta')/(n' - \tilde{n}) > c$. As before, let us compare how the state (θ, n) evolves when starting at each of these points for any given realization of shocks. The difference in the fundamentals is constant at $\tilde{\theta} - \theta'$. Let us check how the difference in n evolves. Remember $P^k_{q'}$ is constructed assuming agents expect others to play according to $\{P^{k-1}_q\}_{q\in\mathcal{Q}}$. Using Lemma 1, we have that, at a given point (θ_t, n_t),

$$\frac{\partial n_t}{\partial t} = \delta\left(\sum_{q\in I_t}\alpha_q - n_t\right), \tag{43}$$

where $I_t = \{q \in \mathcal{Q} : \theta_t > P^{k-1}_q(n_t)\}$. Notice if A is to the right of some threshold P^{k-1}_q (meaning $\theta' > P^{k-1}_q(n')$), so is B (i.e., $\tilde{\theta} > P^{k-1}_q(\tilde{n})$), given that P^{k-1}_q is Lipschitz with constant c while $(\tilde{\theta} - \theta')/(n' - \tilde{n}) > c$. Likewise, whenever B is to the left of some threshold P^{k-1}_q, so is A. It means that at point B the first term in parenthesis in Eq. (43) is always larger than or equal to the same term at point A (i.e., there are at least as many types playing 1 at B than at A). This, plus the fact that $\tilde{n} < n'$, imply $\partial n_0/\partial t$ at A is always smaller than at B, so at $t = 0$, the difference in n is shrinking. Hence, at the earliest future date at which the path that started at B reaches a threshold to the right of $P^k_{q'}$ (if there is any), the path that started at A will still be to the left of this given threshold, meaning the difference in n continues to shrink. Also, the path starting at A will certainly reach a threshold to the left of $P^k_{q'}$ earlier (if there is any), making the difference in n shrink as well. It means that, at any future date, the difference in n between the two paths is always smaller than $n' - \tilde{n}$, so the same calculations used before apply: the difference in terms of flow-payoffs at any future date between the paths starting at B and at A is larger than $d_{q'}(\tilde{\theta} - \theta') - b_{q'}(n' - \tilde{n})$, so we must have $(\tilde{\theta} - \theta')/(n' - \tilde{n}) \le b_{q'}/d_{q'} = c_{q'} \le c$, which is a contradiction. Thus, $\{P^k_q\}_{q\in\mathcal{Q}}$ are Lipschitz with constant c, and so are the limits $\{P^{\infty}_q\}_{q\in\mathcal{Q}}$ (and also $\{P^{\lambda\infty}_q\}_{q\in\mathcal{Q}}$, since they are translations of $\{P^{\infty}_q\}_{q\in\mathcal{Q}}$). □

B.2. Proof of Lemma 2

Suppose $\overline{Z}(n) < \underline{Z}(n)$ for all n in some interval. By Lemma 1, we know that the dynamics of n (whenever in that interval) is given by the following dynamical system:

$$\dot{n}_t = \begin{cases} -\delta n_t & \text{if } \theta_t < \overline{Z}(n_t) \\ \delta(\alpha - n_t) & \text{if } \overline{Z}(n_t) < \theta_t < \underline{Z}(n_t) \\ \delta(1 - n_t) & \text{if } \theta_t > \underline{Z}(n_t). \end{cases}$$

(i) Consider a starting point (θ_0, n_0) such that $\theta_0 = \overline{Z}(n_0)$. With vanishing shocks, that is, $\mu, \sigma \to 0$, since $\underline{Z}(n) > \overline{Z}(n)$, we can focus on the behavior of the system only around $\overline{Z}(n_0)$ to compute $\dot{n}_t$ at $t = 0$ (shocks are not large enough to push θ to the right of $\underline{Z}$). So, in a neighborhood of the threshold $\overline{Z}$, we can write the dynamics of n as:

$$\dot{n}_t = \begin{cases} \delta(\alpha - n_t) & \text{if } \theta_t > \overline{Z}(n_t) \\ -\delta n_t & \text{if } \theta_t < \overline{Z}(n_t). \end{cases}$$

Defining $x_t \equiv n_t/\alpha$, we have that $\dot{x}_t = \dot{n}_t/\alpha$, that is,

$$\dot{x}_t = \begin{cases} \delta(1 - x_t) & \text{if } \theta_t > \overline{Z}(\alpha x_t) \\ -\delta x_t & \text{if } \theta_t < \overline{Z}(\alpha x_t). \end{cases}$$

We can, then, directly apply Theorem 2 in Burdzy et al. (1998), which gives us the desired result. The probability of the system bifurcating up at some point (θ_0, n_0) with $\theta_0 = \overline{Z}(\alpha x_0) \equiv \overline{Z}(n_0)$ and $n_0 < \alpha$ is given by $P(\text{up}) = \frac{\delta(1-x_0)}{\delta(1-x_0)+\delta x_0} = 1 - x_0 = 1 - \frac{n_0}{\alpha}$, $P(\text{down}) = \frac{n_0}{\alpha}$, and the time it takes for the system to bifurcate either up or down converges to zero. If $n_0 > \alpha$, $\dot{n}_0 < 0$ both to the left and to the right of $\overline{Z}(n_0)$, so the system bifurcates down with probability one at time zero.[24]

(ii) As $\mu, \sigma \to 0$, the dynamics around $\underline{Z}$ can be written as:

$$\dot{n}_t = \begin{cases} \delta(1 - n_t) & \text{if } \theta_t > \underline{Z}(n_t) \\ \delta(\alpha - n_t) & \text{if } \theta_t < \underline{Z}(n_t). \end{cases}$$

Define $y_t \equiv \frac{n-\alpha}{1-\alpha}$. $\dot{y}_t = \dot{n}_t/(1-\alpha)$, that is,

$$\dot{y}_t = \begin{cases} \delta(1 - y_t) & \text{if } \theta_t > \underline{Z}((1-\alpha)y_t + \alpha) \\ -\delta(y_t) & \text{if } \theta_t < \underline{Z}((1-\alpha)y_t + \alpha). \end{cases}$$

Applying Theorem 2 in Burdzy et al. (1998), we find that at (θ_0, n_0) with $\theta_0 = \underline{Z}((1-\alpha)y_t + \alpha) \equiv \underline{Z}(n_0)$ and $n_0 > \alpha$, $P(\text{up}) = 1 - y_0 = \frac{1-n_0}{1-\alpha}$, $P(\text{down}) = \frac{n_0-\alpha}{1-\alpha}$ and the time it takes for the system to bifurcate either direction converges to zero. If $n_0 \le \alpha$, then $\dot{n}_0 \ge 0$ both to the right and to the left of $\underline{Z}$, so the system bifurcates up with probability one at time zero.[25] □

B.3. Proof of Proposition 2

(i) Let $\underline{O}(n) > \overline{P}(n)$ $\forall n$. On the equilibrium, agents cannot play strictly dominated strategies, then $\underline{Z}(n) \in [\underline{O}(n), \underline{P}(n)]$ $\forall n$ and $\overline{Z}(n) \in [\overline{O}(n), \overline{P}(n)]$ $\forall n$. Since $[\underline{O}(n), \underline{P}(n)] \cap [\overline{O}(n), \overline{P}(n)] = \emptyset$ $\forall n$, there is no n such that $\underline{Z}(n) = \overline{Z}(n)$.

(ii) Let the dominance regions be such that $\underline{O}(n) \le \overline{P}(n)$ $\forall n \in [0, 1]$. First, notice that it is never the case that $\overline{Z}(n) > \underline{Z}(n)$, for any n. Otherwise, at any point in $(\underline{Z}(n), \overline{Z}(n))$, both types of players would face the same θ, have the same expected path for n_t, and type-$\overline{q}$ would have a higher preference for action 1. Yet, such player would have negative payoff, while a type-$\underline{q}$ player would have positive payoff of playing 1, a contradiction.

Suppose the equilibrium is such that $\overline{Z}(\alpha) < \underline{Z}(\alpha)$. Lemma 2 implies that a type-$\overline{q}$ agent at $\overline{Z}(\alpha)$ expects n_t to decrease with probability one at the maximum rate, while a type-$\underline{q}$ player at $\underline{Z}(\alpha)$ expects n_t to increase with probability one at the maximum rate, which implies $\overline{Z}(\alpha) = \overline{P}(\alpha)$ and $\underline{Z}(\alpha) = \underline{O}(\alpha)$. Thus, $\overline{P}(\alpha) < \underline{O}(\alpha)$, contradiction. We must have that $\overline{Z}(\alpha) = \underline{Z}(\alpha)$. Moreover, this point must be somewhere in the interval $[\underline{O}(\alpha), \overline{P}(\alpha)]$, so that no agent plays strictly dominated strategies in equilibrium.

We also need to show that agents never play the same strategy for every $n \in [0, 1]$. Consider a type-$\underline{q}$ agent at her threshold at $n = 0$. Regardless of the position of the other type's threshold ($\overline{Z}(0) = \underline{Z}(0)$ or $\overline{Z}(0) < \underline{Z}(0)$), the beliefs over n_t such agent hold are the most optimistic as possible, by Lemma 2, and thus we know she is indifferent between both actions exactly at $\underline{O}(0)$. Hence, $\underline{Z}(0) = \underline{O}(0)$. Now, consider the case of type-$\overline{q}$ agents. If they play according to $\underline{O}(0)$, their gains from choosing 1 at this point must be strictly positive, since they have the same expected path for the fundamentals as the low-type agents, the same expected beliefs over the path of n_t and $\overline{\pi}(\theta, n) > \underline{\pi}(\theta, n)$ $\forall(\theta, n)$ by assumption. As we move the candidate to the high-type threshold at $n = 0$ to the left along the θ axis starting at $\underline{O}(0)$, the relative payoff of action 1 decreases for two reasons: the initial θ is smaller, and also the beliefs over n_t become worse. Notice Lemma 2 implies that whenever $\overline{Z}(n) < \underline{Z}(n)$ for $n < \alpha$ we have that $\underline{Z}(n) = \underline{O}(n)$. Then, since thresholds are downward sloping (by Proposition 1), if $\overline{Z}(0) < \underline{Z}(0) = \underline{O}(0)$, it must be that $\underline{Z}(n) = \underline{O}(n)$ for all $n \le \underline{O}^{-1}(\overline{Z}(0))$. It implies that a type-$\overline{q}$ agent at a threshold $\overline{Z}(0) < \underline{Z}(0)$

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[24] Exactly at $n_0 = \alpha$, $\dot{n} = 0$ to the right of $\overline{Z}(n_0)$ and $\dot{n} < 0$ to the left of it, so $P(\text{down})$ is also equal to one.

[25] Exactly at $n_0 = \alpha$, $\dot{n} = 0$ to the left of $\underline{Z}(n_0)$ and $\dot{n} > 0$ to the right of it, so $P(\text{up})$ is also equal to one.

holds the belief that n will bifurcate up with probability 1, but it will increase at a smaller rate (specifically $\dot{n}_t = \delta(\alpha - n_t)$) until it crosses $\underline{O}^{-1}\left(\overline{Z}(0)\right)$, and thereafter it will go up at the maximum rate towards one.[26] We also know that if high-type agents play according to $\overline{O}(0)$, their payoff must be strictly negative, since it would be zero under the most optimistic beliefs possible, which do not hold anymore. Hence, there must be a threshold $\overline{Z}(0)$ in $\left(\overline{O}(0), \underline{O}(0)\right)$ at which the relative payoff of type-$\overline{q}$ agents is zero, i.e., they are indifferent between both actions. Thus, $\overline{Z}(0) < \underline{Z}(0)$. This and the fact that thresholds are downward sloping imply that $\overline{Z}(n) < \underline{Z}(n)$ for all n in some interval $[0, n_1)$. An analogous reasoning implies that $\overline{Z}(n) < \underline{Z}(n)$ for all n in some interval $(n_2, 1]$ as well.

Last, we must show that when the condition on the dominance regions holds with strict inequality, i.e. $\underline{O}(n) < \overline{P}(n)\ \forall n$, $\overline{Z}(n) = \underline{Z}(n)$ for all n in some interval $C \supset \alpha$. Suppose there is no such C. Then, for every arbitrary interval $\tilde{C} \supset \alpha$, $\exists n \in \tilde{C}$ such that $\overline{Z}(n) \neq \underline{Z}(n)$. Fix $\varepsilon > 0$ and let $\hat{C} = [\alpha - \varepsilon, \alpha + \varepsilon]$. There must exist a point $d \in \hat{C}$ such that $\overline{Z}(d) \neq \underline{Z}(d)$. Assume $d > \alpha$.[27] We can choose an appropriate $\hat{\varepsilon} \leq \varepsilon$ in order to write $d = \alpha + \hat{\varepsilon}$. Lemma 2 implies that $\overline{Z}(\alpha + \hat{\varepsilon}) = \overline{P}(\alpha + \hat{\varepsilon})$ and since thresholds are downward sloping we must have that $\overline{Z}(\alpha) \in \left[\overline{P}(\alpha + \hat{\varepsilon}), \overline{P}(\alpha)\right] \equiv A$.[28] Now, consider a $b \in \left[\alpha - \hat{\varepsilon}, \alpha\right]$ and suppose by contradiction that $\overline{Z}(b) \neq \underline{Z}(b)$. Using Lemma 2 once again, we have that $\underline{Z}(b) = \underline{O}(b)$ and, since $\underline{Z}$ is downward sloping, $\underline{Z}(\alpha) \in \left[\underline{O}(\alpha), \underline{O}(b)\right] \equiv B$. Notice that $\underline{Z}(\alpha) = \overline{Z}(\alpha)$ must lie in $A \cap B$. However, if ε is small enough, $A \cap B = \emptyset$, given that $\underline{O}(n) < \overline{P}(n)\ \forall n$, and hence we reach a contradiction. At $n = b$, agents must play according to the same strategy. Finally, since we have fixed an arbitrary b, it must be the case that $\overline{Z}(n) = \underline{Z}(n)$ for all $n \in \left[\alpha - \hat{\varepsilon}, \alpha\right]$, contradicting the fact that $\nexists C \supset \alpha$ such that $\overline{Z}(n) = \underline{Z}(n)\ \forall n \in C$. This concludes the proof. □

B.4. Proof of Proposition 3

Proposition 3 follows from the results in Proposition 6 and the dynamics of n_t presented in the proof of Lemma 1 when $\delta \to \infty$. □

B.5. Proof of Proposition 4

Given a measure u of players playing 1 and some θ, the contribution to potential of increasing $\underline{n}$ in $d\underline{n}$ is $[\pi(\theta, u) + \underline{\varepsilon}](1 - \alpha)d\underline{n}$, and the gain in potential of an increase in $\overline{n}$ by $d\overline{n}$ is $[\pi(\theta, u) + \overline{\varepsilon}]\alpha d\overline{n}$. If $\pi(\theta, u) + \underline{\varepsilon}$ is positive, so is $\pi(\theta, u) + \overline{\varepsilon}$. Also, if $\pi(\theta, u) + \underline{\varepsilon} > 0$, since payoffs are increasing in both arguments we have that $\pi(\theta, u + du) + \underline{\varepsilon} > 0$ for any $du > 0$, hence potential would be maximized by making $\overline{n} = \underline{n} = 1$. Moreover, if $\pi(\theta, u) + \overline{\varepsilon} > 0$, then $\pi(\theta, u + du) + \overline{\varepsilon} > 0$, but it could be that $\pi(\theta, u) + \underline{\varepsilon} < 0$. Another candidate to potential maximizer is then $\overline{n} = 1$ and $\underline{n} = 0$. Also, if $\pi(\theta, u) + \overline{\varepsilon} < 0$, then $\pi(\theta, u) + \underline{\varepsilon} < 0$. Hence, for each θ, potential can only be maximized by having no, all, or only the high-type players choosing action 1. Finding the maximum boils down to comparing these three possibilities:

$$\mathcal{P}(0, 0) = 0, \tag{44}$$

$$\mathcal{P}(1, 0) = \int_0^\alpha \pi(\theta, u)\, du + \alpha\overline{\varepsilon}, \tag{45}$$

$$\mathcal{P}(1, 1) = \int_0^1 \pi(\theta, u)\, du + \alpha\overline{\varepsilon} + (1 - \alpha)\underline{\varepsilon}. \tag{46}$$

Using the definitions from Proposition 6, the dominance regions of interest under vanishing frictions and the payoff structure of Proposition 3 are the curves $\overline{P}(n)$ and $\underline{O}(n)$ satisfying

$$\int_0^n \pi\left(\overline{P}, u\right) du = -n\overline{\varepsilon},$$

$$\int_n^1 \pi\left(\underline{O}, u\right) du = -(1 - n)\underline{\varepsilon}.$$

Small heterogeneity

Lemma 4. *Consider the case where $\overline{P}(n) > \underline{O}(n)\,\forall n$. $(1, 0)$ is never a potential maximizer, i.e.:*

(1) *If $\mathcal{P}(1, 1) > \mathcal{P}(0, 0)$ for a given θ, then $\mathcal{P}(1, 1) > \mathcal{P}(1, 0)$;*
(2) *If $\mathcal{P}(0, 0) \geq \mathcal{P}(1, 1)$ for a given θ, then $\mathcal{P}(0, 0) > \mathcal{P}(1, 0)$.*

Proof. Let $\overline{P}(n) > \underline{O}(n)\,\forall n$.

(1) Suppose, by contradiction, that there exists θ such that $\mathcal{P}(1, 1) > \mathcal{P}(0, 0)$, but $\mathcal{P}(1, 1) \leq \mathcal{P}(1, 0)$. Using (44)–(46), we have that

$$\mathcal{P}(1, 1) > \mathcal{P}(0, 0) \Longleftrightarrow \int_0^1 \pi(\theta, u)\, du > -\alpha\overline{\varepsilon} - (1 - \alpha)\underline{\varepsilon}, \tag{47}$$

and

$$\mathcal{P}(1, 0) \geq \mathcal{P}(1, 1) \iff \int_\alpha^1 \pi(\theta, u)\, du \leq -(1 - \alpha)\underline{\varepsilon}. \tag{48}$$

The definition of dominance regions implies that, in particular,

$$\int_0^\alpha \pi\left(\overline{P}(\alpha), u\right) du = -\alpha\overline{\varepsilon}, \tag{49}$$

$$\int_\alpha^1 \pi\left(\underline{O}(\alpha), u\right) du = -(1 - \alpha)\underline{\varepsilon}. \tag{50}$$

From (48) and (50), we have

$$\int_\alpha^1 \pi(\theta, u)\, du \leq \int_\alpha^1 \pi\left(\underline{O}(\alpha), u\right) du \Longrightarrow \theta \leq \underline{O}(\alpha).$$

From (47), (49) and (50), we have

$$\int_0^1 \pi(\theta, u)\, du > \int_0^\alpha \pi\left(\overline{P}(\alpha), u\right) du + \int_\alpha^1 \pi\left(\underline{O}(\alpha), u\right) du > \int_0^1 \pi\left(\underline{O}(\alpha), u\right) du \Longrightarrow \theta > \underline{O}(\alpha),$$

a contradiction.

(2) Now, suppose by contradiction that there exists θ such that $\mathcal{P}(0, 0) \geq \mathcal{P}(1, 1)$, but $\mathcal{P}(0, 0) \leq \mathcal{P}(1, 0)$. Using (44)–(46), we have that

$$\mathcal{P}(0, 0) \geq \mathcal{P}(1, 1) \Longleftrightarrow \int_0^1 \pi(\theta, u)\, du \leq -\alpha\overline{\varepsilon} - (1 - \alpha)\underline{\varepsilon}, \tag{51}$$

and

$$\mathcal{P}(0, 0) \leq \mathcal{P}(1, 0) \iff \int_0^\alpha \pi(\theta, u)\, du + \alpha\overline{\varepsilon} \geq 0. \tag{52}$$

From (49) and (52), we have

$$\int_0^\alpha \pi(\theta, u)\, du \geq \int_0^\alpha \pi\left(\overline{P}(\alpha), u\right) du \Longrightarrow \theta \geq \overline{P}(\alpha).$$

[26] Notice that in the limiting case with $\mu, \sigma \to 0$, this analysis can be done regardless of the position of equilibrium thresholds for higher values of n, since thresholds are downward sloping and θ remains almost constant.

[27] This is without loss of generality since an analogous argument holds if $d < \alpha$.

[28] Remember we had before that $\overline{Z}(\alpha) = \underline{Z}(\alpha) \in \left[\underline{O}(\alpha), \overline{P}(\alpha)\right]$.

From (49)–(51), we have

$$\int_0^1 \pi(\theta, u)\, du \le \int_0^\alpha \pi\left(\overline{P}(\alpha), u\right) du + \int_\alpha^1 \pi\left(\underline{O}(\alpha), u\right) du$$
$$< \int_0^1 \pi\left(\overline{P}(\alpha), u\right) du \Longrightarrow \theta < \overline{P}(\alpha),$$

a contradiction. □

Hence, the potential maximizer when $\overline{P}(n) > \underline{O}(n)\,\forall n$ is given by

$$\arg\max \mathcal{P}\left(\overline{n}, \underline{n}; \theta\right) = \begin{cases} (0,0) & \text{for } \theta < \hat{z}^*, \\ \{(0,0),(1,1)\} & \text{for } \theta = \hat{z}^*, \\ (1,1) & \text{for } \theta > \hat{z}^*, \end{cases} \quad (53)$$

where, as before, $\hat{z}^*$ satisfies

$$\int_0^1 \pi\left(\hat{z}^*, u\right) du = -\alpha\overline{\varepsilon} - (1-\alpha)\underline{\varepsilon} \equiv -\hat{\varepsilon}.$$

Now, consider the case where $\overline{P}(n) = \underline{O}(n)$ for all n. The same steps followed in the proof of Lemma 4 can be used to show that $\mathcal{P}(1,1) > \mathcal{P}(0,0) \Longrightarrow \mathcal{P}(1,1) > \mathcal{P}(1,0)$, and that $\mathcal{P}(0,0) > \mathcal{P}(1,1) \Longrightarrow \mathcal{P}(0,0) > \mathcal{P}(1,0)$. When $\mathcal{P}(0,0;\theta) = \mathcal{P}(1,1;\theta)$, which happens if and only if $\theta = \hat{z}^*$, we have that $\mathcal{P}(0,0;\hat{z}^*) = \mathcal{P}(1,0;\hat{z}^*)$. This result can be shown as follows. Summing (49) and (50) and using $\overline{P}(\alpha) = \underline{O}(\alpha)$, we get that $\hat{z}^* = \overline{P}(\alpha) = \underline{O}(\alpha)$, and using (49) one can see that $\mathcal{P}(1,0;\overline{P}(\alpha)) = 0$. Thus, the potential maximizer is given by

$$\arg\max \mathcal{P}\left(\overline{n}, \underline{n}; \theta\right) = \begin{cases} (0,0) & \text{for } \theta < \hat{z}^*, \\ \{(1,1),(0,0),(1,0)\} & \text{for } \theta = \hat{z}^*, \\ (1,1) & \text{for } \theta > \hat{z}^*. \end{cases}$$

The only difference from Eq. (53) is that $(1,0)$ is also a maximizer when $\theta = \hat{z}^*$.

Large heterogeneity

Consider the case where $\overline{P}(n) < \underline{O}(n)\,\forall n$. From (52) we know that $\mathcal{P}(1,0) \ge \mathcal{P}(0,0)$ if and only if $\theta \ge \overline{z}^* = \overline{P}(\alpha)$. From (48) we know $\mathcal{P}(1,1) \ge \mathcal{P}(1,0)$ if and only if $\theta \ge \underline{z}^* = \underline{O}(\alpha)$. Since $\underline{z}^* = \underline{O}(\alpha) > \overline{P}(\alpha) = \overline{z}^*$ in the case of large heterogeneity, the potential maximizer is given by

$$\arg\max \mathcal{P}(\overline{n}, \underline{n}; \theta) = \begin{cases} (0,0) & \text{for } \theta < \overline{z}^*, \\ \{(0,0),(1,0)\} & \text{for } \theta = \overline{z}^*, \\ (1,0) & \text{for } \theta \in (\overline{z}^*, \underline{z}^*) \\ \{(1,0),(1,1)\} & \text{for } \theta = \underline{z}^*, \\ (1,1) & \text{for } \theta > \underline{z}^*. \end{cases} \quad \square$$

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